REFLEXIVITY AND RIGIDITY FOR COMPLEXES, II: SCHEMES

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ABSTRACT. We prove basic facts about reflexivity in derived categories over noetherian schemes; and about related notions such as semidualizing complexes, invertible complexes, and Gorenstein-perfect maps. Also, we study a notion of rigidity with respect to semidualizing complexes, in particular, relative dualizing complexes for Gorenstein-perfect maps. Our results include theorems of Yekutieli and Zhang concerning rigid dualizing complexes on schemes. This work is a continuation of part I, which dealt with commutative rings.

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Introduction

This paper is concerned with properties of complexes over noetherian schemes, that play important roles in duality theory. Some such properties, like (derived) reflexivity, have been an integral part of the theory since its inception; others, like rigidity, appeared only recently. Our main results reveal new aspects of such concepts and establish novel links between them.

Similar questions over commutative rings were examined in [5]. Additional topics treated there are semidualizing complexes, complexes of finite Gorenstein dimension, perfect complexes, invertible complexes, and rigidity with respect to semidualizing complexes, as well as versions of these notions relative to essentially-finite-type ring-homomorphisms that have finite flat dimension or, more generally, finite Gorenstein dimension. In this sequel we globalize such considerations, that is, extend them to the context of schemes.

This work is a substantial application of Grothendieck duality theory, seen as the study of a twisted inverse image pseudofunctor $(-)^!$ defined on appropriate categories of schemes. Duality theory provides interpretations of the local facts, a technology to globalize them, and suggestions for further directions of development.

To place our work in context, we review two methods for proving existence of $(-)^!$ for noetherian schemes and separated scheme-maps of finite type. The original approach of Grothendieck involves the construction of a 'coherent family' of dualizing complexes; details are presented in [15] and revised in [10]. An alternative method, based on Nagata compactifications and sketched in [12] and [23], is developed in [19]. Recent extensions of these approaches to maps essentially of finite type provide a principal object of this study—the concept of rigidity—and one of our main tools.

Indeed, rigid dualizing complexes over rings, introduced by Van den Bergh [22] in the context of non-commutative algebraic geometry, are used by Yekutieli and Zhang [25, 26] in an ongoing project aiming to simplify Grothendieck's construction of $(-)^!$, and extend it to schemes essentially of finite type over a regular ring of finite Krull dimension. On the other hand, Nayak [21] proved an analog of Nagata's compactification theorem and extended the pseudofunctor $(-)^!$ to the category of all noetherian schemes and their separated maps essentially of finite type. We work in this category.

Next we describe in some detail the notions and results of the paper. Comparison with earlier work is postponed until the end of this Introduction.

Let X be a scheme, $\mathsf{D}(X)$ the derived category of the category of $\mathcal{O}_X$-modules, and $\mathsf{D}^{\mathsf{b}}_{\mathsf{c}}(X) \subset \mathsf{D}(X)$ the full subcategory whose objects are the complexes with coherent homology that vanishes in all but finitely many degrees. For F and A in $\mathsf{D}(X)$, we say that F is *derived A-reflexive* if both F and $\mathsf{R}\mathcal{H}om_X(F, A)$ are in $\mathsf{D}^{\mathsf{b}}_{\mathsf{c}}(X)$, and if the canonical $\mathsf{D}(X)$-map is an isomorphism

$$F \xrightarrow{\sim} \mathsf{R}\mathcal{H}om_X\big(\mathsf{R}\mathcal{H}om_X(F, A), A\big).$$

When $\mathcal{O}_X$ itself is derived A-reflexive the complex A is said to be *semidualizing*. (The classical notion of *dualizing complex* includes the additional requirement that A be isomorphic, in $\mathsf{D}(X)$, to a bounded complex of injective sheaves.)

In Chapter 1 we prove basic results about semidualizing complexes in $\mathsf{D}(X)$, and examine their interplay with perfect complexes, that is, complexes $F \in \mathsf{D}^{\mathsf{b}}_{\mathsf{c}}(X)$

such that for every $x \in X$ the stalk F_x is isomorphic in $\mathsf{D}(\mathcal{O}_{X,x})$ to a bounded complex of flat $\mathcal{O}_{X,x}$-modules (or equivalently, such that F is isomorphic in $\mathsf{D}(X)$ to a bounded complex of flat $\mathcal{O}_X$-modules).

In Chapter 2 we explore conditions on a scheme-map $f\colon X \to Y$ that allow for the transfer of properties, such as reflexivity, along standard functors $\mathsf{D}(Y) \to \mathsf{D}(X)$.

One such condition involves the notion of *perfection relative to* f, defined for F in $\mathsf{D}^{\mathsf{b}}_{\mathsf{c}}(X)$ by replacing $\mathcal{O}_{X,x}$ with $\mathcal{O}_{Y,f(x)}$ in the definition of perfection. If this condition holds with $F = \mathcal{O}_X$, then f is said to be perfect. Flat maps are classical examples. We relate the basic global notions to ones that are local not only in the Zariski topology, but also in the flat topology; that is, we find that they behave rather well under faithfully flat maps. (This opens the way to examination of more general sites, not undertaken here.) As a sample of results concerning ascent and descent along perfect maps, we quote from Theorem 2.2.5 and Corollary 2.2.6:

Theorem 1. *Let $f\colon X \to Y$ be a perfect map and B a complex in $\mathsf{D}^{+}_{\mathsf{c}}(Y)$.*

If $M \in \mathsf{D}(Y)$ is derived B-reflexive, then the complex $\mathsf{L}f^*M$ in $\mathsf{D}(X)$ is both derived $\mathsf{L}f^*B$-reflexive and derived $f^!B$-reflexive. For $M = \mathcal{O}_Y$ this says that if B is semidualizing then so are $\mathsf{L}f^*B$ and $f^!B$.

For each of these four statements, the converse holds if M and B are in $\mathsf{D}^{\mathsf{b}}_{\mathsf{c}}(Y)$, and f is faithfully flat, or f is perfect, proper and surjective.

The perfection of f can be recognized by its *relative dualizing complex*, $f^!\mathcal{O}_Y$. Indeed, f is perfect if and only if $f^!\mathcal{O}_Y$ is relatively perfect. Furthermore, if f is perfect, then every perfect complex in $\mathsf{D}(X)$ is derived $f^!\mathcal{O}_Y$-reflexive.

One sees, in particular, that when f is perfect the complex $f^!\mathcal{O}_Y$ is semidualizing. We take this condition as the definition of *G-perfect* maps. (Here G stands for Gorenstein.) They form a class significantly larger than that of perfect maps. For instance, when the scheme Y is Gorenstein *every* scheme map $X \to Y$ is G-perfect. In §2.3, we prove some basic properties of such maps, and, more generally, of $\mathcal{O}_X$-complexes that are derived $f^!\mathcal{O}_Y$-reflexive. For such complexes there exist nice dualities with respect to the relative dualizing complex (see Corollary 2.3.12).

Quasi-Gorenstein maps are defined by the condition that $f^!\mathcal{O}_Y$ is perfect. A very special case has been extensively studied: a *flat* map is quasi-Gorenstein if and only if all its fibers are Gorenstein schemes. On the other hand, *every* map of Gorenstein schemes is quasi-Gorenstein. Every quasi-Gorenstein map is G-perfect.

All these classes interact in many pleasing ways with composition and base change of scheme-maps, as explicated mainly in §2.5. Such results generalize, and often strengthen, theorems about ascent and descent along perfect maps. For example, several assertions in Theorem 1 are obtained by taking $f = \mathrm{id}^X$ in the following theorem, which is proved as part of Proposition 2.5.6:

Theorem 2. *Let $Z \xrightarrow{g} X \xrightarrow{f} Y$ be scheme-maps, with f quasi-Gorenstein.*

The composition fg is G-perfect if and only if so is g.

Also, if g is quasi-Gorenstein then so is fg.

In Chapter 3 we define *rigidity* with respect to an arbitrary semidualizing complex $A \in \mathsf{D}(X)$. An A-rigid structure on F in $\mathsf{D}^{\mathsf{b}}_{\mathsf{c}}(X)$ is a $\mathsf{D}(X)$-isomorphism

$$\rho\colon F \xrightarrow{\ \sim\ } \mathsf{R}\mathcal{H}om_X(\mathsf{R}\mathcal{H}om_X(F,A),F).$$

We say that (F,ρ) is an *A-rigid pair*; $F \in \mathsf{D}^{\mathsf{b}}_{\mathsf{c}}(X)$ is an *A-rigid complex* if such an isomorphism ρ exists. Morphisms of rigid pairs are defined in the obvious way.

In Theorem 3.1.7 we establish the basic fact about rigid pairs:

Theorem 3. *Let A be a semidualizing complex in $\mathsf{D}(X)$.*

For each quasi-coherent $\mathcal{O}_X$-ideal I such that $I^2 = I$, there exists a canonical A-rigid structure on IA; and every A-rigid pair is uniquely isomorphic in $\mathsf{D}(X)$ to such an IA along with its canonical structure.

The theorem validates the term 'rigid', as it implies that the only automorphism of a rigid pair is the identity. It also shows that isomorphism classes of A-rigid complexes correspond bijectively to the open-and-closed subsets of X. A more precise description—in terms of those subsets—of the skeleton of the category of rigid pairs appears in Theorem 3.2.6.

In the derived category, gluing over open coverings is usually not possible; but it is for idempotent ideals (Proposition C.8). Consequently the uniqueness expressed by Theorem 3 leads to gluing for rigid pairs, in the following strong sense:

Theorem 4. *For any open cover (U_α) of X and family (F_α, ρ_α) of $A|_{U_\alpha}$-rigid pairs such that for all α, α' the restrictions of (F_α, ρ_α) and $(F_{\alpha'}, \rho_{\alpha'})$ to $U_\alpha \cap U_{\alpha'}$ are isomorphic, there is a unique (up to unique isomorphism) A-rigid pair (F, ρ), such that for each α, $(F, \rho)|_{U_\alpha} \simeq (F_\alpha, \rho_\alpha)$.*

This gluing property holds even under the flat topology, see Theorem 3.2.9.

In §3.3 we study complexes that are *relatively rigid*, that is, rigid with respect to the relative dualizing complex $f^!\mathcal{O}_Y$ of a G-perfect map $f\colon X \to Y$ (a complex that is, by the definition of such maps, semidualizing). As a consequence of gluing for rigid complexes under the flat topology, gluing for relatively rigid complexes holds under the étale topology, see Proposition 3.3.1.

Relative rigidity behaves naturally with respect to (G-)perfect maps, in the sense that certain canonical isomorphisms from duality theory, involving relative dualizing complexes, respect the additional rigid structure. In Corollary 3.3.5 we show that, when g is perfect, the twisted inverse image functor $g^!$ preserves relative rigidity; and also, for a composition $Z \xrightarrow{g} X \xrightarrow{f} Y$ where f is G-perfect, we demonstrate the interaction of rigidity with the canonical isomorphism

$$g^!\mathcal{O}_X \otimes^{\mathsf{L}}_Z \mathsf{L}g^* f^!\mathcal{O}_Y \xrightarrow{\ \sim\ } (fg)^!\mathcal{O}_Y\,.$$

In Corollary 3.3.7 we do the same with respect to flat base change. Such results are obtained as applications of simple necessary and sufficient condition for additive functors of rigid complexes to be liftable to rigid pairs, detailed in Theorem 3.3.2.

The results above can be applied to complete some work started in [6]. In that paper, we associated a relative dualizing complex to each essentially-finite-type homomorphism of commutative rings, but did not touch upon the functoriality properties of that complex. This aspect of the construction can now be supplied by using the fact that the sheafification of the complex in [6] is a relative dualizing complex for the corresponding map of spectra; see Example 2.3.2. One can then use the results in §3.3, discussed above, to enrich the reduction isomorphism [6, 4.1] to a functorial one. For such applications, it is crucial to work with scheme-maps that are *essentially* of finite type; this is one of our reasons for choosing this category in the setup for this paper.

Notions and notation related to scheme-maps, as well as pertinent material from Grothendieck duality theory, as used in this paper, are surveyed in the appendices.

We finish the introduction by reviewing connections to earlier work.

The results in Chapter 1 are, for the most part, extensions to the global situation of results proved over commutative rings in [5]; the transfer is fairly straightforward.

Homomorphisms of commutative noetherian rings that track Gorenstein-type properties were introduced and studied in [2, 3, 17], without finiteness hypotheses. Those papers are based on Auslander and Bridger's [1] theory of Gorenstein dimension, which is defined in terms of resolutions by finite modules or projective modules, and so does not globalize. The scheme-maps defined and studied in Chapter 2 are based on a different description of finite Gorenstein dimension for ring-homomorphisms essentially of finite type, obtained in [5, 2.2].

The developments in Chapter 3 are largely motivated and inspired by work of Yekutieli and Zhang, starting with [24]. One of their goals was to construct a new foundation for Grothendieck duality theory. Making extensive use of differential graded algebras (DGAs), in [25, 26] they extended Van den Bergh's construction [22] of rigid *dualizing* complexes to schemes *essentially of finite type over a regular ring of finite Krull dimension*, and analyzed the behavior of such complexes under some types of perfect maps. Theirs is a novel approach, especially with regard to the introduction of DGAs into the subject. However, it remains to be seen whether, once all the details are fully exposed, it will prove to be simpler than the much more generally applicable theory presented, for example, in [19].

We come to rigidity from the opposite direction, presupposing duality theory and making no use of DGAs. The concept obtained in this way applies to *semi*dualizing complexes over arbitrary schemes, and behaves well under *all* perfect scheme-maps. In the setup of [26], the regularity of the base ring implies that relative dualizing complexes are actually dualizing. To compare results, one also needs to know that, when both apply, our concept of rigidity coincides with Yekutieli and Zhang's. This follows from the Reduction Theorem [6, 4.1]; see [5, 8.5.5].

1. Derived reflexivity over schemes

Rings are assumed to be commutative, and both rings and schemes are assumed to be noetherian.

1.1. **Standard homomorphisms.** Let $(X, \mathcal{O}_X)$ be a scheme and $\mathsf{D}(X)$ the derived category of the category of sheaves of $\mathcal{O}_X$-modules.

Let $\mathsf{D}^+(X)$, resp. $\mathsf{D}^-(X)$, be the full subcategory of $\mathsf{D}(X)$ having as objects those complexes whose cohomology vanishes in all but finitely many negative, resp. positive, degrees; set $\mathsf{D}^{\mathsf{b}}(X) := \mathsf{D}^+(X) \cap \mathsf{D}^-(X)$. For $\bullet = +, -$ or b, let $\mathsf{D}^\bullet_{\mathsf{c}}(X)$, resp. $\mathsf{D}^\bullet_{\mathsf{qc}}(X)$, be the full subcategory of $\mathsf{D}(X)$ with objects those complexes all of whose cohomology sheaves are *coherent,* resp. *quasi-coherent*.

To lie in $\mathsf{D}^\bullet_*(X)$ ($* = \mathsf{c}$ or qc, and $\bullet = +, -$ or b) is a *local condition*: if (U_α) is an open cover of X, then $F \in \mathsf{D}(X)$ lies in $\mathsf{D}^\bullet_*(X)$ if and only if for all α the restriction $F|_{U_\alpha}$ lies in $\mathsf{D}^\bullet_*(U_\alpha)$.

A number of canonical homomorphisms play a fundamental role in this paper.

Remark 1.1.1. There is a standard trifunctorial isomorphism, relating the derived tensor and sheaf-homomorphism functors (see e.g., [19, §2.6]):

$$(1.1.1.1)\quad \mathsf{R}\mathcal{H}om_X\big(E \otimes^{\mathsf{L}}_X F, G\big) \xrightarrow{\sim} \mathsf{R}\mathcal{H}om_X\big(E, \mathsf{R}\mathcal{H}om_X(F, G)\big) \quad (E, F, G \in \mathsf{D}(X))$$

from which one gets, by application of the composite functor $\mathrm{H}^0\mathsf{R}\Gamma(X,-)$,

$$\mathrm{Hom}_{\mathsf{D}(X)}\big(E\otimes^{\mathsf{L}}_X F, G\big) \xrightarrow{\sim} \mathrm{Hom}_{\mathsf{D}(X)}\big(E, \mathsf{R}\mathcal{H}om_X(F,G)\big). \tag{1.1.1.2}$$

The map corresponding via (1.1.1.2) to the identity map of $\mathsf{R}\mathcal{H}om_X(F,G)$

$$\varepsilon=\varepsilon^F_G\colon \mathsf{R}\mathcal{H}om_X(F,G)\otimes^{\mathsf{L}}_X F\to G \qquad (F,G\in\mathsf{D}(X)) \tag{1.1.1.3}$$

is called *evaluation*. When F is a flat complex in $\mathsf{D}^-(X)$ (or more generally, any q-flat complex in $\mathsf{D}(X)$, see [19, §2.5]), and G is an injective complex in $\mathsf{D}^+(X)$ (or more generally, any q-injective complex in $\mathsf{D}(X)$, see [19, §2.3]), one verifies that ε is induced by the family of maps of complexes

$$\varepsilon(U)\colon\ \mathrm{Hom}_{\mathcal{O}_X(U)}(F(U),G(U))\otimes_{\mathcal{O}_X(U)} F(U)\to G(U) \qquad (U\subseteq X \text{ open})$$

where, for homogeneous $\alpha\in\mathrm{Hom}_{\mathcal{O}_X(U)}(F(U),G(U))$ and $b\in F(U)$,

$$\varepsilon(U)(\alpha\otimes b)=\alpha(b).$$

Basic properties of supports of complexes are recalled for further reference.

Remark 1.1.2. For any $F\in\mathsf{D}(X)$, the *support of* F is the set

$$\mathrm{Supp}_X F := \{\, x\in X \mid H^n(F_x)\neq 0 \text{ for some } n\,\}. \tag{1.1.2.1}$$

If $F\in\mathsf{D}^{\mathsf{b}}_{\mathsf{c}}(X)$, then $\mathrm{Supp}_X F$ is a *closed subset* of X. Also, for all F and G in $\mathsf{D}^-_{\mathsf{c}}(X)$, it follows from e.g., [5, A.6] that

$$\mathrm{Supp}_X(F\otimes^{\mathsf{L}}_X G)=\mathrm{Supp}_X F\cap\mathrm{Supp}_X G. \tag{1.1.2.2}$$

Note that $\mathrm{Supp}_X F=\emptyset$ if and only if $F=0$ in $\mathsf{D}(X)$.

The following example opens the door to applications of the results in [5].

Example 1.1.3. Let R be a ring. Let $\mathsf{D}(R)$ be the derived category of the category of R-modules, and define, as above, its full subcategories $\mathsf{D}^{\bullet}(R)$ for $\bullet=\mathsf{+}$, $\mathsf{-}$ or b. Let $\mathsf{D}^{\bullet}_{\mathsf{f}}(R)$ be the full subcategory of $\mathsf{D}^{\bullet}(R)$ having as objects those complexes whose cohomology modules are all *finite*, i.e., finitely generated, over R.

For the affine scheme $X=\mathrm{Spec}\,R$, the functor that associates to each complex $M\in\mathsf{D}(R)$ its sheafification $M^\sim$ is an *equivalence of categories* $\mathsf{D}^{\bullet}_{\mathsf{f}}(R)\xrightarrow{\approx}\mathsf{D}^{\bullet}_{\mathsf{c}}(X)$, see [7, 5.5]; when $\bullet=\mathsf{+}$ or b, see also [15, p. 133, 7.19].

There is a natural bifunctorial isomorphism

$$(M\otimes^{\mathsf{L}}_R N)^\sim \xrightarrow{\sim} M^\sim\otimes^{\mathsf{L}}_X N^\sim \qquad \big(M,\, N\in\mathsf{D}(R)\big); \tag{1.1.3.1}$$

to define it one may assume that M and N are suitable flat complexes, so that $\otimes^{\mathsf{L}}$ becomes ordinary $\otimes$, see [19, §2.5 and (2.6.5)].

There is also a natural bifunctorial map

$$\mathsf{R}\mathrm{Hom}_R(M,N)^\sim \longrightarrow \mathsf{R}\mathcal{H}om_X\big(M^\sim,\, N^\sim\big), \tag{1.1.3.2}$$

defined to be the one that corresponds via (1.1.1.2) to the composite map

$$\mathsf{R}\mathrm{Hom}_R(M,N)^\sim\otimes^{\mathsf{L}}_X M^\sim \xrightarrow{\sim} (\mathsf{R}\mathrm{Hom}_R(M,N)\otimes^{\mathsf{L}}_R M)^\sim \xrightarrow{\varepsilon^\sim} N^\sim,$$

where the isomorphism comes from (1.1.3.1), and the *evaluation map* ε corresponds to the identity map of $\mathsf{R}\mathrm{Hom}_R(M,N)$ via the analog of (1.1.1.2) over $\mathsf{D}(R)$.

The map (1.1.3.2) is an *isomorphism* if $M\in\mathsf{D}^-_{\mathsf{f}}(R)$ and $N\in\mathsf{D}^+(R)$. To show this for variable M and fixed N one can use the "way-out" Lemma [15, p. 68, 7.1], with A the opposite of the category of R-modules and P the family $(R^n)_{n>0}$, to reduce to the case $M=R$, where, one checks, the map is the obvious isomorphism.

1.2. **Derived multiplication by global functions.** Let $(X, \mathcal{O}_X)$ be a scheme. Here we discuss some technicalities about the natural action of $\mathrm{H}^0(X, \mathcal{O}_X)$ on $\mathsf{D}(X)$.

We identify $\mathrm{H}^0(X, \mathcal{O}_X)$ with $\mathrm{Hom}_{\mathsf{D}(X)}(\mathcal{O}_X, \mathcal{O}_X)$ via the ring isomorphism that takes $\alpha \in \mathrm{H}^0(X, \mathcal{O}_X)$ to multiplication by α. For $\alpha \in \mathrm{H}^0(X, \mathcal{O}_X)$ and $F \in \mathsf{D}(X)$, let $\mu_F(\alpha)$ ("multiplication by α in F") be the natural composite $\mathsf{D}(X)$-map

$$F \simeq \mathcal{O}_X \otimes^{\mathsf{L}}_X F \xrightarrow{\alpha \otimes^{\mathsf{L}}_X 1} \mathcal{O}_X \otimes^{\mathsf{L}}_X F \simeq F,$$

or equivalently,

$$F \simeq F \otimes^{\mathsf{L}}_X \mathcal{O}_X \xrightarrow{1 \otimes^{\mathsf{L}}_X \alpha} F \otimes^{\mathsf{L}}_X \mathcal{O}_X \simeq F.$$

Clearly, for any $\mathsf{D}(X)$-map $\phi\colon F \to C$,

$$\phi\alpha := \phi \circ \mu_F(\alpha) = \mu_C(\alpha) \circ \phi =: \alpha\phi.$$

Furthermore, using the obvious isomorphism $(\mathcal{O}_X \otimes^{\mathsf{L}}_X F)[1] \xrightarrow{\sim} \mathcal{O}_X \otimes^{\mathsf{L}}_X F[1]$ one sees that $\mu_F(\alpha)$ commutes with translation, that is, $\mu_F(\alpha)[1] = \mu_{F[1]}(\alpha)$.

Thus, the family $(\mu_F)_{F \in \mathsf{D}(X)}$ maps $\mathrm{H}^0(X, \mathcal{O}_X)$ into the ring C_X consisting of endomorphisms of the identity functor of $\mathsf{D}(X)$ that commute with translation—the *center* of $\mathsf{D}(X)$. It is straightforward to verify that this map is an injective ring homomorphism onto the subring of *tensor-compatible* members of C_X, that is, those $\eta \in \mathsf{C}_X$ such that for all F, $G \in \mathsf{D}(X)$,

$$\eta(F \otimes^{\mathsf{L}}_X G) = \eta(F) \otimes^{\mathsf{L}}_X \mathrm{id}^G = \mathrm{id}^G \otimes^{\mathsf{L}}_X \eta(G).$$

The category $\mathsf{D}(X)$ is C_X-linear: for all F, $G \in \mathsf{D}(X)$, $\mathrm{Hom}_{\mathsf{D}(X)}(F, G)$ has a natural structure of C_X-module, and composition of maps is C_X-bilinear. So $\mathsf{D}(X)$ is also $\mathrm{H}^0(X, \mathcal{O}_X)$-linear, via μ.

Lemma 1.2.1. *For any F, $G \in \mathsf{D}(X)$ and $\mathsf{D}(X)$-homomorphism$\alpha\colon \mathcal{O}_X \to \mathcal{O}_X$, and $\mu_\bullet(\alpha)$ as above, there are equalities*

$$\mathsf{R}\mathcal{H}om_X(\mu_F(\alpha), G) = \mu_{\mathsf{R}\mathcal{H}om_X(F,G)}(\alpha) = \mathsf{R}\mathcal{H}om_X(F, \mu_G(\alpha))).$$

Proof. Consider, for any $E \in \mathsf{D}(X)$, the natural trifunctorial isomorphism

$$\tau\colon \mathrm{Hom}_{\mathsf{D}(X)}(E \otimes^{\mathsf{L}}_X F, G) \xrightarrow{\sim} \mathrm{Hom}_{\mathsf{D}(X)}(E, \mathsf{R}\mathcal{H}om_X(F, G)).$$

From tensor-compatibility in the image of μ, and $\mathrm{H}^0(X, \mathcal{O}_X)$-linearity of $\mathsf{D}(X)$, it follows that for any $\alpha \in \mathrm{H}^0(X, \mathcal{O}_X)$, the map $\mu_E(\alpha)$ induces multiplication by α in both the source and target of τ. Functoriality shows then that τ is an isomorphism of $\mathrm{H}^0(X, \mathcal{O}_X)$-modules.

Again, tensor-compatibility implies that $\mu_F(\alpha)$ induces multiplication by α in the source of the $\mathrm{H}^0(X, \mathcal{O}_X)$-linear map τ, hence also in the target. Thus, by functoriality, $\mathsf{R}\mathcal{H}om_X(\mu_F(\alpha), G)$ induces multiplication by α in the target of τ. For $E = \mathsf{R}\mathcal{H}om_X(F, G)$, this gives $\mathsf{R}\mathcal{H}om_X(\mu_F(\alpha), G) = \mu_{\mathsf{R}\mathcal{H}om_X(F,G)}(\alpha)$. One shows similarly that $\mathsf{R}\mathcal{H}om_X(F, \mu_G(\alpha)) = \mu_{\mathsf{R}\mathcal{H}om_X(F,G)}(\alpha)$. ∎

1.3. **Derived reflexivity.** Let $(X, \mathcal{O}_X)$ be a scheme.

One has, for all A and F in $\mathsf{D}(X)$, a *biduality morphism*

$$\delta^A_F\colon F \to \mathsf{R}\mathcal{H}om_X\big(\mathsf{R}\mathcal{H}om_X(F, A), A\big), \tag{1.3.0.0}$$

corresponding via (1.1.1.2) to the natural composition

$$F \otimes^{\mathsf{L}}_X \mathsf{R}\mathcal{H}om_X(F, A) \xrightarrow{\sim} \mathsf{R}\mathcal{H}om_X(F, A) \otimes^{\mathsf{L}}_X F \xrightarrow{\varepsilon^F_A} A.$$

The map δ_F^A "commutes" with restriction to open subsets (use [19, 2.4.5.2]).

When A is a q-injective complex in $\mathsf{D}(X)$, δ_F^A is induced by the family

$$\delta(U)\colon F(U) \to \mathrm{Hom}_{\mathcal{O}_X(U)}(\mathrm{Hom}_{\mathcal{O}_X(U)}(F(U), A(U)), A(U)) \qquad (U \subseteq X \text{ open})$$

of maps of complexes, where, for each $n \in F(U)$ of degree b, the map $\delta(U)(n)$ is

$$\alpha \mapsto (-1)^{ab}\alpha(n)\,,$$

for $\alpha \in \mathrm{Hom}_{\mathcal{O}_X(U)}(F(U), A(U))$ homogeneous of degree a.

Definition 1.3.1. Given A and F in $\mathsf{D}(X)$, we say that F is *derived A-reflexive* if both F and $\mathsf{R}\mathcal{H}om_R(F, A)$ are in $\mathsf{D}^{\mathsf{b}}_{\mathsf{c}}(X)$ and δ_F^A is an isomorphism.

This is a *local condition*: for any open cover (U_α) of X, F is derived A-reflexive if and only if the same is true over every U_α for the restrictions of F and A. Also, as indicated below, if U is affine, say $U := \mathrm{Spec}\, R$, and $C, M \in \mathsf{D}^{\mathsf{b}}_{\mathsf{f}}(R)$, then $M^\sim$ is derived $C^\sim$-reflexive in $\mathsf{D}(U) \iff M$ is derived C-reflexive in $\mathsf{D}(R)$.

Example 1.3.2. When $X = \mathrm{Spec}\, R$ and $M, C \in \mathsf{D}(R)$, it follows that with δ_M^C as in [5, (2.0.1)], the map $\delta_{M^\sim}^{C^\sim}$ factors naturally as

$$M^\sim \xrightarrow{(\delta_M^C)^\sim} \big(\mathsf{R}\mathrm{Hom}_R(\mathsf{R}\mathrm{Hom}_R(M, C), C)\big)^\sim \xrightarrow{\ s\ } \mathsf{R}\mathcal{H}om_X\big(\mathsf{R}\mathcal{H}om_X(M^\sim, C^\sim), C^\sim\big),$$

where, as in (1.1.3.2), the map s is an isomorphism if $M \in \mathsf{D}^{-}_{\mathsf{f}}(R)$, $C \in \mathsf{D}^{+}(R)$ and $\mathsf{R}\mathrm{Hom}_R(M, C) \in \mathsf{D}^{\mathsf{b}}_{\mathsf{f}}(R)$. Thus, derived reflexivity globalizes the notion in [5, §2].

From [5, 2.1 and 2.3] one now gets:

Proposition 1.3.3. *Let X be a noetherian scheme, and A, $F \in \mathsf{D}^{\mathsf{b}}_{\mathsf{c}}(X)$. Then the following conditions are equivalent.*

(i) *F is derived A-reflexive.*

(ii) *$\mathsf{R}\mathcal{H}om_X(F, A) \in \mathsf{D}^{-}(X)$ and there exists an isomorphism in $\mathsf{D}(X)$*

$$F \xrightarrow{\ \sim\ } \mathsf{R}\mathcal{H}om_X\big(\mathsf{R}\mathcal{H}om_X(F, A), A\big).$$

(iii) *$\mathsf{R}\mathcal{H}om_X(F, A)$ is derived A-reflexive and $\mathrm{Supp}_X F \subseteq \mathrm{Supp}_X A$.* □

Remark 1.3.4. For $A = \mathcal{O}_X$ the theorem above shows that $F \in \mathsf{D}^{\mathsf{b}}_{\mathsf{c}}(X)$ is derived $\mathcal{O}_X$-reflexive if and only if so is $\mathsf{R}\mathcal{H}om_X(F, \mathcal{O}_X)$.

In the affine case, $X = \mathrm{Spec}\, R$, for any $M \in \mathsf{D}^{\mathsf{b}}_{\mathsf{f}}(R)$, the derived $\mathcal{O}_X$-reflexivity of $M^\sim$ is equivalent to finiteness of the Gorenstein dimension of M, as defined by Auslander and Bridger; see [1].

Definition 1.3.5. An $\mathcal{O}_X$-complex A is *semidualizing* if $\mathcal{O}_X$ is derived A-reflexive. In other words, $A \in \mathsf{D}^{\mathsf{b}}_{\mathsf{c}}(X)$ and the map $\chi^A\colon \mathcal{O}_X \to \mathsf{R}\mathcal{H}om_X(A, A)$ corresponding via (1.1.1.2) to the natural map $\mathcal{O}_X \otimes^{\mathsf{L}}_X A \to A$ is an isomorphism.

As above, this condition is local on X. When $X = \mathrm{Spec}\, R$, a complex $C \in \mathsf{D}^{\mathsf{b}}_{\mathsf{f}}(R)$ is semidualizing in the commutative-algebra sense (that is, R is derived C-reflexive, see e.g., [5, §3]) if and only if $C^\sim$ is semidualizing in the present sense.

Lemma 1.3.6. *If $A \in \mathsf{D}(X)$ is semidualizing then each $\mathsf{D}(X)$-endomorphism of A is multiplication by a uniquely determined $\alpha \in \mathrm{H}^0(X, \mathcal{O}_X)$.*

Proof. With $\chi^A\colon \mathcal{O}_X \to \mathsf{R}\mathcal{H}om_X(A,A)$ as in Definition 1.3.5, the map μ_A is easily seen to factor as follows:

$$\begin{aligned}\operatorname{Hom}_{\mathsf{D}(X)}(\mathcal{O}_X,\mathcal{O}_X) \xrightarrow{\text{via }\chi^A} &\operatorname{Hom}_{\mathsf{D}(X)}\big(\mathcal{O}_X,\mathsf{R}\mathcal{H}om_X(A,A)\big)\\ &\cong \operatorname{Hom}_{\mathsf{D}(X)}(\mathcal{O}_X\otimes^{\mathsf{L}}_X A, A)\\ &\cong \operatorname{Hom}_{\mathsf{D}(X)}(A,A).\end{aligned}$$

The assertion results. □

Lemma 1.3.7. *Let X be a noetherian scheme. If A is a semidualizing $\mathcal{O}_X$-complex, then* $\operatorname{Supp}_X A = X$. *Furthermore, If there is an isomorphism $A \simeq A_1 \oplus A_2$ then* $\operatorname{Supp}_X A_1 \cap \operatorname{Supp}_X A_2 = \emptyset$.

Proof. The $\mathcal{O}_X$-complex $\mathsf{R}\mathcal{H}om_X(A,A)$, which is isomorphic in $\mathsf{D}(X)$ to $\mathcal{O}_X$, is acyclic over the open set $X \setminus \operatorname{Supp}_X A$. This implies $\operatorname{Supp}_X A = X$.

As to the second assertion, taking stalks at arbitrary $x \in X$ reduces the problem to showing that if R is a local ring, and M_1 and M_2 in $\mathsf{D}(R)$ are such that the natural map

$$R \to \mathsf{R}\mathrm{Hom}_R(M_1\oplus M_2, M_1\oplus M_2) = \bigoplus\nolimits_{i,j=1}^{2} \mathsf{R}\mathrm{Hom}_R(M_i,M_j)$$

is an isomorphism, then either $M_1 = 0$ or $M_2 = 0$.

But clearly, R being local, at most one of the direct summands $\mathsf{R}\mathrm{Hom}_R(M_i,M_j)$ can be nonzero, so for $i=1$ or $i=2$ the identity map of M_i is 0, whence the conclusion. □

1.4. **Perfect complexes.** Again, $(X,\mathcal{O}_X)$ is a scheme.

Definition 1.4.1. An $\mathcal{O}_X$-complex P is *perfect* if X is a union of open subsets U such that the restriction $P|_U$ is $\mathsf{D}(U)$-isomorphic to a bounded complex of finite-rank locally free $\mathcal{O}_U$-modules.

From [16, p. 115, 3.5 and p. 135, 5.8.1], one gets:

Remark 1.4.2. The complex P is perfect if and only if $P \in \mathsf{D_c}(X)$ and P is isomorphic in $\mathsf{D}(X)$ to a bounded complex of flat $\mathcal{O}_X$-modules.

Perfection is a local condition. If $X = \operatorname{Spec} R$ and $M \in \mathsf{D}(R)$ then $M^\sim$ is perfect if and only if N is isomorphic in $\mathsf{D}(R)$ to a bounded complex of finite projective R-modules, cf. [5, §4]. The next result is contained in [8, 2.1.10]; see also [5, 4.1].

Theorem 1.4.3. *$P \in \mathsf{D}^{\mathsf{b}}_{\mathsf{c}}(X)$ is perfect if and only if so is $\mathsf{R}\mathcal{H}om_X(P,\mathcal{O}_X)$.* □

Proposition 1.4.4. *Let A and P be in $\mathsf{D}(X)$, with P perfect.*

If $F \in \mathsf{D}(X)$ is derived A-reflexive then so is $P \otimes^{\mathsf{L}}_X F$; in particular, P is derived $\mathcal{O}_X$-reflexive. If A is semidualizing then P is derived A-reflexive.

Proof. The assertion being local, we may assume that P is a bounded complex of finite-rank free $\mathcal{O}_X$-modules. If two vertices of a triangle are derived A-reflexive then so is the third, whence an easy induction on the number of degrees in which P is nonzero shows that if F is A-reflexive then so is $P \otimes^{\mathsf{L}}_X F$. To show that P is derived $\mathcal{O}_X$-reflexive, take $A = \mathcal{O}_X = F$.

For the final assertion, take $F = \mathcal{O}_X$. □

A partial converse is given by the next result:

Theorem 1.4.5. *Let $F \in \mathsf{D_c}(X)$, let $A \in \mathsf{D_c^+}(X)$, and let P be a perfect $\mathcal{O}_X$-complex with $\operatorname{Supp}_X P \supseteq \operatorname{Supp}_X F$. If $P \otimes^{\mathsf{L}}_X F$ is in $\mathsf{D^b_c}(X)$, or $P \otimes^{\mathsf{L}}_X F$ is perfect, or $P \otimes^{\mathsf{L}}_X F$ is derived A-reflexive, then the corresponding property holds for F.*

Proof. The assertions are all local, and the local statements are proved in [5, 4.3, 4.4, and 4.5], modulo sheafification; see Example 1.1.3. □

We'll need the following isomorphisms (cf. [16, pp. 152–153, 7.6 and 7.7]).

Let E, F and G be complexes in $\mathsf{D}(X)$, and consider the map

$$\mathsf{R}\mathcal{H}om_X(E,F) \otimes^{\mathsf{L}}_X G \to \mathsf{R}\mathcal{H}om_X(E, F \otimes^{\mathsf{L}}_X G), \tag{1.4.5.1}$$

corresponding via (1.1.1.2) to the natural composition

$$(\mathsf{R}\mathcal{H}om_X(E,F) \otimes^{\mathsf{L}}_X G) \otimes^{\mathsf{L}}_X E \xrightarrow{\sim} (\mathsf{R}\mathcal{H}om_X(E,F) \otimes^{\mathsf{L}}_X E) \otimes^{\mathsf{L}}_X G \xrightarrow{\varepsilon \otimes^{\mathsf{L}}_X 1} F \otimes^{\mathsf{L}}_X G.$$

where ε is the evaluation map from (1.1.1.3).

Lemma 1.4.6. *Let E, F and G be complexes in $\mathsf{D}(X)$.*

(1) *When either E or G is perfect, the map (1.4.5.1) is an isomorphism*

$$\mathsf{R}\mathcal{H}om_X(E,F) \otimes^{\mathsf{L}}_X G \simeq \mathsf{R}\mathcal{H}om_X(E, F \otimes^{\mathsf{L}}_X G).$$

(2) *When G is perfect, there is a natural isomorphism*

$$\mathsf{R}\mathcal{H}om_X(E \otimes^{\mathsf{L}}_X G, F) \simeq \mathsf{R}\mathcal{H}om_X(E,F) \otimes^{\mathsf{L}}_X \mathsf{R}\mathcal{H}om_X(G, \mathcal{O}_X).$$

Proof. (1). Whether the map (1.4.5.1) is an isomorphism is a local question, so if E is perfect then one may assume that E is a bounded complex of finite-rank free $\mathcal{O}_X$-modules. The affirmative answer is then given by a simple induction on the number of degrees in which E is nonzero.

A similar argument applies when G is perfect.

(2). Setting $\check{G} := \mathsf{R}\mathcal{H}om_X(G, \mathcal{O}_X)$, we get from (1), with (E, F, G) changed to $(G, \mathcal{O}_X, F)$, an isomorphism

$$F \otimes^{\mathsf{L}}_X \check{G} \simeq \check{G} \otimes^{\mathsf{L}}_X F \xrightarrow{\sim} \mathsf{R}\mathcal{H}om_X(G, F).$$

This induces the second isomorphism below:

$$\begin{aligned} \mathsf{R}\mathcal{H}om_X(E \otimes^{\mathsf{L}}_X G, F) &\xrightarrow{\sim} \mathsf{R}\mathcal{H}om_X(E, \mathsf{R}\mathcal{H}om_X(G,F)) \\ &\xrightarrow{\sim} \mathsf{R}\mathcal{H}om_X(E, F \otimes^{\mathsf{L}}_X \check{G}) \\ &\xrightarrow{\sim} \mathsf{R}\mathcal{H}om_X(E,F) \otimes^{\mathsf{L}}_X \check{G}; \end{aligned}$$

the isomorphism comes from (1.1.1.1) and the third from (1), since $\check{G}$ is also perfect, by Theorem 1.4.3. The desired map is the composite isomorphism. □

1.5. **Invertible complexes.** Again, $(X, \mathcal{O}_X)$ is a scheme.

Definition 1.5.1. A complex in $\mathsf{D}(X)$ is *invertible* if it is semidualizing and perfect.

This condition is local. If $X = \operatorname{Spec} R$ and $M \in \mathsf{D}(R)$, then M is invertible in the sense of [5, §5] if and only if $M^{\sim}$ is invertible in the present sense.

Recall that Σ denotes the usual translation (suspension) operator on complexes.

Theorem 1.5.2. *For $L \in \mathsf{D^b_c}(X)$ the following conditions are equivalent.*

(i) *L is invertible.*

(ii) *$L^{-1} := \mathsf{R}\mathcal{H}om_X(L, \mathcal{O}_X)$ is invertible.*

(iii) *Each $x \in X$ has an open neighborhood U_x such that for some integer r_x, there is a $\mathsf{D}(U_x)$-isomorphism $L|_{U_x} \simeq \Sigma^{r_x}\mathcal{O}_{U_x}$.*

(iii′) *For each connected component U of X there is an integer r, a locally free rank-one $\mathcal{O}_U$-module $\mathcal{L}$, and a $\mathsf{D}(U)$-isomorphism $L|_U \simeq \Sigma^r\mathcal{L}$.*

(iv) *For some $F \in \mathsf{D_c}(X)$ there is an isomorphism $F \otimes^{\mathsf{L}}_X L \simeq \mathcal{O}_X$.*

(v) *For all $G \in \mathsf{D}(X)$ the evaluation map ε from* (1.1.1.3) *is an isomorphism*

$$\mathsf{R}\mathcal{H}om_X(L,G) \otimes^{\mathsf{L}}_X L \xrightarrow{\sim} G.$$

(v′) *For all G and G' in $\mathsf{D}(X)$, the natural composite map* (see 1.1.1.1)

$$\begin{aligned}\mathsf{R}\mathcal{H}om_X(G' \otimes^{\mathsf{L}}_X L, G) \otimes^{\mathsf{L}}_X L &\xrightarrow{\sim} \mathsf{R}\mathcal{H}om_X(L \otimes^{\mathsf{L}}_X G', G) \otimes^{\mathsf{L}}_X L \\ &\xrightarrow{\sim} \mathsf{R}\mathcal{H}om_X(L, \mathsf{R}\mathcal{H}om_X(G',G)) \otimes^{\mathsf{L}}_X L \\ &\xrightarrow[\varepsilon]{} \mathsf{R}\mathcal{H}om_X(G',G)\end{aligned}$$

is an isomorphism.

Proof. When (i) holds, Lemma 1.4.6(2), with $E = \mathcal{O}_X$ and $G = L = F$, yields:

$$\mathcal{O}_X \xrightarrow{\sim} \mathsf{R}\mathcal{H}om_X(L,L) \xrightarrow{\sim} L \otimes^{\mathsf{L}}_X L^{-1}. \tag{1.5.2.1}$$

(i) $\Leftrightarrow$ (ii). By Theorem 1.4.3, the $\mathcal{O}_X$-complex L is perfect if and only if so is L^{-1}. If (i) holds, then (1.5.2.1), Proposition 1.4.4 (with $A = \mathcal{O}_X = F$, $P = L$), and Lemma 1.4.6(1) give isomorphisms

$$\mathcal{O}_X \xrightarrow{\sim} L \otimes^{\mathsf{L}}_X L^{-1} \xrightarrow{\sim} \mathsf{R}\mathcal{H}om_X(L^{-1}, \mathcal{O}_X) \otimes^{\mathsf{L}}_X L^{-1} \xrightarrow{\sim} \mathsf{R}\mathcal{H}om_X(L^{-1}, L^{-1}),$$

so that by Proposition 1.3.3(ii) (with $F = \mathcal{O}_X$ and $A = L^{-1}$), the $\mathcal{O}_X$-module L^{-1} is semidualizing; since it also perfect (ii) holds.

The same argument with L and L^{-1} interchanged establishes that (ii) $\Rightarrow$ (i).

(i) $\Rightarrow$ (iii). One may assume here that X is affine. Then, since L is invertible, [5, 5.1] gives that the stalk at x of the cohomology of L vanishes in all but one degree, where it is isomorphic to $\mathcal{O}_{X,x}$. The cohomology of L is bounded and coherent, therefore there is an open neighborhood U_x of x over which the cohomology of L vanishes in all but one degree, where it is isomorphic to $\mathcal{O}_{U_x}$, i.e., (iii) holds.

(iii) $\Rightarrow$ (iv). If (iii) holds then the evaluation map (1.1.1.3) (with $A = L$ and $G = \mathcal{O}_X$) is an isomorphism $L^{-1} \otimes^{\mathsf{L}}_X L \xrightarrow{\sim} \mathcal{O}_X$.

(iv) $\Rightarrow$ (i). This is a local statement that is established (along with some other unstated equivalences) in [5, 5.1]; see also [13, 4.7].

(iii) $\Rightarrow$ (iii′). The function $x \mapsto r_x$ must be locally constant, so of constant value, say r, on U; and then in $\mathsf{D}(U)$, $L \simeq \Sigma^r H^{-r}(L)$.

(iii′) $\Rightarrow$ (iii). This implication is clear.

(i) $\Rightarrow$ (v). The first of the following isomorphisms comes from Lemma 1.4.6(2) (with $(E,F,G) = (L,G,\mathcal{O}_X)$), and the second from (1.5.2.1):

$$\mathsf{R}\mathcal{H}om_X(L,G) \otimes^{\mathsf{L}}_X L \xrightarrow{\sim} L^{-1} \otimes^{\mathsf{L}}_X G \otimes^{\mathsf{L}}_X L \xrightarrow{\sim} G.$$

That this composite isomorphism is ε is essentially the definition of the isomorphism

$$L^{-1} \otimes^{\mathsf{L}}_X G = \mathsf{R}\mathcal{H}om_X(L, \mathcal{O}_X) \otimes^{\mathsf{L}}_X G \xrightarrow{\sim} \mathsf{R}\mathcal{H}om_X(L,G)\,;$$

see the proof of Lemma 1.4.6.

(v) $\Rightarrow$ (iv). Set $F := L^{-1}$, and apply (v) for $G = \mathcal{O}_X$.

(v) $\Leftrightarrow$ (v′). Replace G in (v) by $\mathsf{R}\mathcal{H}om_X(G',G)$; or G' in (v′) by $\mathcal{O}_X$. $\square$

Corollary 1.5.3. *Let L_1 and L_2 be complexes in $\mathsf{D_c}(X)$.*

(1) *If* L_1 *and* L_2 *are invertible, then so is* $L_1 \otimes^{\mathsf{L}}_X L_2$.
(2) *If* L_1 *is in* $\mathsf{D}^{\mathsf{b}}_{\mathsf{c}}(X)$ *and* $L_1 \otimes^{\mathsf{L}}_X L_2$ *is invertible, then* L_1 *is invertible.*
(3) *For any scheme-map* $g\colon Z \to X$, *if* L_1 *is invertible then so is* $\mathsf{L}g^*L_1$.

Proof. For (1), use Theorem 1.5.2(iii′); for (2), Theorem 1.5.2(iv)—noting that the F there may be taken to be the invertible complex L^{-1}, and that tensoring with an invertible complex takes $\mathsf{D}_{\mathsf{c}}(X)$ into itself; and for (3), the fact that g maps any connected component of Z into a connected component of X. □

Corollary 1.5.4. *Let* A, L *and* F *be complexes in* $\mathsf{D}^{\mathsf{b}}_{\mathsf{c}}(X)$, *with* L *invertible.*
(1) F *is derived* A*-reflexive if and only if it is derived* $L \otimes^{\mathsf{L}}_X A$*-reflexive.*
(2) F *is derived* A*-reflexive if and only if* $F \otimes^{\mathsf{L}}_X L$ *is derived* A*-reflexive.*
(3) A *is semidualizing if and only if* $L \otimes^{\mathsf{L}}_X A$ *is semidualizing.*

Proof. From, say, Theorem 1.5.2(iii′) and Lemma 1.4.6(1), one gets

$$\mathsf{R}\mathcal{H}om_X(F, A) \in \mathsf{D}^{\mathsf{b}}_{\mathsf{c}}(X) \iff \mathsf{R}\mathcal{H}om_X(F, L \otimes^{\mathsf{L}}_X A) \in \mathsf{D}^{\mathsf{b}}_{\mathsf{c}}(X).$$

Since $L^{-1} \otimes^{\mathsf{L}}_X L \simeq \mathcal{O}_X$, (1) follows now from Lemma 1.4.6; (2) follows from Theorem 1.5.2(iii); and (3) follows from (1). □

Remark 1.5.5. A complex $A \in \mathsf{D}^{\mathsf{b}}_{\mathsf{c}}(X)$ is *pointwise dualizing* if every $F \in \mathsf{D}^{\mathsf{b}}_{\mathsf{c}}(X)$ is derived A-reflexive (see [5, 6.2.2]). Such an A is semidualizing: take $F = \mathcal{O}_X$.

It is proved in [5, 8.3.1] that $\mathcal{O}_X$ *is pointwise dualizing if and only if* X *is a Gorenstein scheme* (i.e., the local ring $\mathcal{O}_{X,x}$ is Gorenstein for all $x \in X$).

It follows from [5, 5.7] that invertible complexes can be characterized as those that are semidualizing and derived $\mathcal{O}_X$-reflexive. Hence *when* X *is Gorenstein,* $A \in \mathsf{D}^{\mathsf{b}}_{\mathsf{c}}(X)$ *is semidualizing* $\iff$ A *is pointwise dualizing* $\iff$ A *is invertible.*

2. Gorenstein-type properties of scheme-maps

All schemes are assumed to be noetherian; all scheme-maps are assumed to be essentially of finite type (*see Appendix* A) *and separated.*

2.1. **Perfect maps.** Let $f\colon X \to Y$ be a scheme-map.

Let $f_0\colon X \to Y$ denote the underlying map of topological spaces, and f_0^{-1} the left adjoint of the direct image functor f_{0*} from sheaves of abelian groups on X to sheaves of abelian groups on Y. There is then a standard way of making $f_0^{-1}\mathcal{O}_Y$ into a sheaf of commutative rings on X, whose stalk at any point $x \in X$ is $\mathcal{O}_{Y,f(x)}$.

Definition 2.1.1. An $\mathcal{O}_X$-complex F is *perfect relative to* f—or, as we will write, *perfect over* f—if it is in $\mathsf{D}^{\mathsf{b}}_{\mathsf{c}}(X)$, and in the derived category of the category of $f_0^{-1}\mathcal{O}_Y$-modules F is isomorphic to a bounded complex of flat $f_0^{-1}\mathcal{O}_Y$-modules.

The map f is *perfect* if $\mathcal{O}_X$ is perfect over f. (See [16, p. 250, Définition 4.1].)

Remark 2.1.2. Since f is essentially of finite type, there is always such a U for which $f|U$ factors as (essentially smooth) ∘ (closed immersion). If $X \xrightarrow{i} W \to Y$ is such a factorization, then F is perfect over f if and only if i_*F is perfect over id^W: the proof of [16, pp. 252, 4.4] applies here (see Remark A.3).

Using [16, p. 242, 3.3], one sees that perfection over f is local on X, in the sense that F has this property if and only if every $x \in X$ has an open neighborhood U such that $F|_U$ is perfect over $f|_U$.

Perfection over id^X is equivalent to perfection in $\mathsf{D}(X)$; see Remark 1.4.2.

Let $\mathsf{P}(f)$ be the full subcategory of $\mathsf{D}(X)$ whose objects are all the complexes that are perfect over f; and let $\mathsf{P}(X) := \mathsf{P}(\mathrm{id}^X)$ be the full subcategory of $\mathsf{D}(X)$ whose objects are all the perfect $\mathcal{O}_X$-complexes.

Example 2.1.3. If $f\colon X = \operatorname{Spec} S \to \operatorname{Spec} K = Y$ corresponds to a homomorphism of noetherian rings $\sigma\colon K \to S$, then $\mathsf{P}(f)$ is equivalent to the full subcategory $\mathsf{P}(\sigma) \subseteq \mathsf{D}^{\mathsf{b}}_{\mathsf{f}}(S)$ with objects those complexes M that are isomorphic in $\mathsf{D}(K)$ to some bounded complex of flat K-modules; this follows from [16, p. 168, 2.2.2.1 and p. 242, 3.3], in view of the standard equivalence, given by sheafification, between finite S-modules and coherent $\mathcal{O}_X$-modules.

Recall that an exact functor $\mathsf{F}\colon \mathsf{D}(Y) \to \mathsf{D}(X)$ is said to be *bounded below* if there is an integer d such that for all $M \in \mathsf{D}(Y)$ and $n \in \mathbb{Z}$ the following holds:

$$H^i(M) = 0 \ \text{ for all } i < n \implies H^j(\mathsf{F}(M)) = 0 \ \text{ for all } j < n - d,$$

By substituting ">" for "<" in the preceding definition one obtains the notion of *bounded above*. If F is bounded below (resp. bounded above) then, clearly, $\mathsf{F}\mathsf{D}^{+}(Y) \subseteq \mathsf{D}^{+}(Y)$ (resp. $\mathsf{F}\mathsf{D}^{-}(Y) \subseteq \mathsf{D}^{-}(Y)$).

Remark 2.1.4. For *every* scheme-map f the functor $\mathsf{L}f^*$ is bounded above. It is bounded below if and only if f is perfect. When f is perfect, one has

$$\mathsf{L}f^*\mathsf{D}^{\mathsf{b}}_{\mathsf{c}}(Y) \subseteq \mathsf{D}^{\mathsf{b}}_{\mathsf{c}}(X)\,.$$

For, $\mathsf{L}f^*$ is bounded above and below, hence, as above, $\mathsf{L}f^*\mathsf{D}^{\mathsf{b}}(Y) \subseteq \mathsf{D}^{\mathsf{b}}(X)$. Also, $\mathsf{L}f^*\mathsf{D}_{\mathsf{c}}(Y) \subseteq \mathsf{D}_{\mathsf{c}}(X)$, see [15, p. 99, 4.4], whose proof uses [15, p. 73, 7.3] and compatibility of $\mathsf{L}f^*$ with open base change to reduce to the assertion that $\mathsf{L}f^*\mathcal{O}_Y = \mathcal{O}_X$.

The following characterization of perfection of f, in terms of the twisted inverse image functor $f^!$, was proved for finite-type f in [19, 4.9.4] and then extended to the essentially finite-type case in [21, 5.9].

Remark 2.1.5. For any scheme-map $f\colon X \to Y$, and for all M, N in $\mathsf{D}^{+}_{\mathsf{qc}}(Y)$, there is defined in [19, §4.9] and [21, 5.7–5.8] a functorial $\mathsf{D}(X)$-map

$$f^!M \otimes^{\mathsf{L}}_X \mathsf{L}f^*N \to f^!(M \otimes^{\mathsf{L}}_Y N). \tag{2.1.5.1}$$

The following conditions on f are equivalent:

(i) The map f is perfect.
(ii) The functor $f^!\colon \mathsf{D}^{+}_{\mathsf{qc}}(Y) \to \mathsf{D}^{+}_{\mathsf{qc}}(X)$ is bounded above and below.
(iii) The complex $f^!\mathcal{O}_Y$ is perfect over f.
(iv) When M is perfect, $f^!M$ is perfect over f; and whenever $M \otimes^{\mathsf{L}}_Y N$ is in $\mathsf{D}^{+}_{\mathsf{qc}}(Y)$, natural the map (2.1.5.1) is an isomorphism

$$f^!M \otimes^{\mathsf{L}}_X \mathsf{L}f^*N \xrightarrow{\ \sim\ } f^!(M \otimes^{\mathsf{L}}_Y N). \tag{2.1.5.2}$$

From (ii) one gets, as above, $f^!\mathsf{D}^{\mathsf{b}}_{\mathsf{qc}}(Y) \subseteq \mathsf{D}^{\mathsf{b}}(X)$; and the last paragraph in §5.4 of [20] gives

$$f^!\mathsf{D}^{+}_{\mathsf{c}}(Y) \subseteq \mathsf{D}^{+}_{\mathsf{c}}(X). \tag{2.1.5.3}$$

Thus, for perfect f, one has

$$f^!\mathsf{D}^{\mathsf{b}}_{\mathsf{c}}(Y) \subseteq \mathsf{D}^{\mathsf{b}}_{\mathsf{c}}(X). \tag{2.1.5.4}$$

Remark 2.1.6. If $f\colon X \to Y$ is a perfect map, then each complex that is perfect over f (in particular, $\mathcal{O}_X$) is derived $f^!\mathcal{O}_Y$-reflexive.

This is given by [16, p. 259, 4.9.2], in whose proof "smooth" can be replaced by "essentially smooth," see [6, 5.1].

Next we establish some further properties of perfect maps for later use.

Lemma 2.1.7. *Let $f\colon X \to Y$ be a scheme-map, and M, B complexes in $\mathsf{D}(Y)$.*

If f is an open immersion, or if f is perfect, M is in $\mathsf{D}^-_{\mathsf{c}}(Y)$ and B is in $\mathsf{D}^+_{\mathsf{qc}}(Y)$, then there are natural isomorphisms

$$\mathsf{L}f^*\mathsf{R}\mathcal{H}om_Y(M,B) \xrightarrow{\sim} \mathsf{R}\mathcal{H}om_X(\mathsf{L}f^*M, \mathsf{L}f^*B)\,, \tag{2.1.7.1}$$

$$f^!\mathsf{R}\mathcal{H}om_Y(M,B) \xrightarrow{\sim} \mathsf{R}\mathcal{H}om_X(\mathsf{L}f^*M, f^!B)\,. \tag{2.1.7.2}$$

Proof. As a map in $\mathsf{D}(X)$, (2.1.7.1) comes from B.1.5. To show it an isomorphism we may assume Y affine, say $Y = \operatorname{Spec} R$. Then by [7, 5.5] and [15, p. 42, 4.6.1 (dualized)], any $M \in \mathsf{D}^-_{\mathsf{c}}(Y)$ is isomorphic to the sheafification of a complex of finite-rank free R-modules, vanishing in all sufficiently large degrees; so [19, p. 181, (4.6.7)] gives the desired assertion.

For (2.1.7.2), use [19, 4.2.3(e)] when f is proper; and then in the general case, compactify, see Appendix A. □

Remark 2.1.8. Let $f\colon X \to Y$ be a perfect and proper scheme map.

One has $\mathsf{R}f_*(\mathsf{D}^{\mathsf{b}}_{\mathsf{c}}(X)) \subseteq \mathsf{D}^{\mathsf{b}}_{\mathsf{c}}(Y)$, by [16, p. 237, 2.2.1]. Moreover, if $F \in \mathsf{D}^{\mathsf{b}}_{\mathsf{c}}(X)$ is perfect, then so is $\mathsf{R}f_*F$; see Remark 1.4.2 and [16, p. 250, Proposition 3.7.2].

Remark 2.1.9. In $\mathsf{D}(X)$ there is a natural map

$$\alpha(E,F,G)\colon \mathsf{R}\mathcal{H}om_X(E,F) \to \mathsf{R}\mathcal{H}om_X(E \otimes^{\mathsf{L}}_X G, F \otimes^{\mathsf{L}}_X G) \qquad \big(E,F,G \in \mathsf{D}(X)\big),$$

corresponding via (1.1.1.2) to

$$(\mathsf{R}\mathcal{H}om_X(E,F) \otimes^{\mathsf{L}}_X E) \otimes^{\mathsf{L}}_X G \xrightarrow{\varepsilon \otimes^{\mathsf{L}}_X \mathbf{1}} F \otimes^{\mathsf{L}}_X G$$

where ε is evaluation (1.1.1.3).

Assume now that f is perfect. By Remark 2.1.5 there is a natural isomorphism

$$\mathsf{L}f^*N \otimes^{\mathsf{L}}_X f^!\mathcal{O}_Y \simeq f^!N \qquad (N \in \mathsf{D}^+_{\mathsf{qc}}(Y)). \tag{2.1.9.1}$$

Hence $\alpha(\mathsf{L}f^*M, \mathsf{L}f^*N, f^!\mathcal{O}_Y)$ gives rise to a natural map, for all $M, N \in \mathsf{D}^+_{\mathsf{qc}}(Y)$,

$$\beta(M,N,f)\colon \mathsf{R}\mathcal{H}om_X(\mathsf{L}f^*M, \mathsf{L}f^*N) \to \mathsf{R}\mathcal{H}om_X(f^!M, f^!N). \tag{2.1.9.2}$$

Lemma 2.1.10. *When $f\colon X \to Y$ is perfect, M is in $\mathsf{D}^{\mathsf{b}}_{\mathsf{c}}(Y)$, and N is in $\mathsf{D}^+_{\mathsf{qc}}(Y)$, the map $\beta(M,N,f)$ is an isomorphism.*

Proof. One checks, using §B.4, §B.3(i), and Lemma 2.1.7, that the question is local on both X and Y. Hence, via [15, p. 133, 7.19], one may assume that M is a complex of finite-rank free $\mathcal{O}_Y$-modules.

By Remarks 2.1.4 and 2.1.5, respectively, the functors $\mathsf{L}f^*$ and $f^!$ are bounded both above and below. Therefore, for every fixed N in $\mathsf{D}^+_{\mathsf{qc}}(Y)$, the source and target of $\beta(M,N,f)$ are bounded below functors of M. Now one can argue as in the proof of [15, p.69, (iv)] to reduce the problem to the case $M = \mathcal{O}_Y$. For this M, one uses a similar argument to reduce to the case where also $N = \mathcal{O}_Y$ holds.

One checks that $\beta(\mathcal{O}_Y, \mathcal{O}_Y, f)$ is the canonical map $\mathcal{O}_X \to \mathsf{R}\mathcal{H}om_X(f^!\mathcal{O}_Y, f^!\mathcal{O}_Y)$, a map that, by Remark 2.1.6, is an isomorphism. □

Lemma 2.1.11. *Let $f\colon X \to Y$ be a perfect map*

When M is in $\mathsf{D}^-_{\mathsf{c}}(Y)$ and B is in $\mathsf{D}^+_{\mathsf{c}}(Y)$, the complex $\mathsf{L}f^*M$ is derived $\mathsf{L}f^*B$-reflexive if and only if it is derived $f^!B$-reflexive.

Proof. We deal first with the boundedness conditions in Definition 1.3.1. The condition $\mathsf{L}f^*M \in \mathsf{D}^{\mathsf{b}}_{\mathsf{c}}(X)$ holds throughout, by assumption.

Assume that $\mathsf{R}\mathcal{H}om_X(\mathsf{L}f^*M, \mathsf{L}f^*B)$ is in $\mathsf{D}^{\mathsf{b}}_{\mathsf{c}}(X)$. As $\mathsf{R}\mathcal{H}om_Y(M,B) \in \mathsf{D}^+_{\mathsf{c}}(Y)$ (see [15, p. 92, 3.3]), one gets from Remark 2.1.5 and (2.1.7.1) an isomorphism

$$f^!\mathcal{O}_Y \otimes^{\mathsf{L}}_X \mathsf{R}\mathcal{H}om_X(\mathsf{L}f^*M, \mathsf{L}f^*B) \simeq f^!\mathsf{R}\mathcal{H}om_Y(M,B)\,. \tag{2.1.11.1}$$

By Remark 2.1.5(iii), $f^!\mathcal{O}_Y \in \mathsf{D}^{\mathsf{b}}_{\mathsf{c}}(X)$, so it follows that $f^!\mathsf{R}\mathcal{H}om_Y(M,B) \in \mathsf{D}^-_{\mathsf{c}}(X)$. On the other hand, by (2.1.5.3), $f^!\mathsf{R}\mathcal{H}om_Y(M,B) \in \mathsf{D}^+_{\mathsf{c}}(X)$. We conclude that $f^!\mathsf{R}\mathcal{H}om_Y(M,B) \in \mathsf{D}^{\mathsf{b}}_{\mathsf{c}}(X)$, and so by (2.1.7.2), that $\mathsf{R}\mathcal{H}om_X(\mathsf{L}f^*M, f^!B) \in \mathsf{D}^{\mathsf{b}}_{\mathsf{c}}(X)$.

Suppose, conversely, that $\mathsf{R}\mathcal{H}om_X(\mathsf{L}f^*M, f^!B) \in \mathsf{D}^{\mathsf{b}}_{\mathsf{c}}(X)$, so that by (2.1.7.2), there is an integer n such that

$$H^i\big(f^!\mathsf{R}\mathcal{H}om_Y(M,B)\big) = 0 \quad \text{for all} \quad i > n.$$

Using (2.1.7.1) and Remark 2.1.4 one gets:

$$\mathsf{R}\mathcal{H}om_X(\mathsf{L}f^*M, \mathsf{L}f^*B) \simeq \mathsf{L}f^*\mathsf{R}\mathcal{H}om_Y(M,B) \in \mathsf{D}_{\mathsf{c}}(X).$$

Also, $f^!\mathcal{O}_Y \in \mathsf{D}^{\mathsf{b}}_{\mathsf{c}}(X)$, by Remark 2.1.5, and it follows from an application of (i)–(iii) in B.3 to a local factorization of f as (essentially smooth)$\circ$(closed immersion)—or from Proposition 2.3.9—that $\operatorname{Supp}_X f^!\mathcal{O}_Y = X$. So except for the trivial case where X is empty, there is an integer m such that

$$H^m f^!\mathcal{O}_Y \neq 0 \quad \text{and} \quad H^j f^!\mathcal{O}_Y = 0 \quad \text{for all} \quad j > m.$$

Hence, by (2.1.11.1), for each x in X and for all all $k > n - m$, [5, A.4.3] gives $(H^k\mathsf{R}\mathcal{H}om_X(\mathsf{L}f^*M, \mathsf{L}f^*B))_x = 0$. It follows that $\mathsf{R}\mathcal{H}om_X(\mathsf{L}f^*M, \mathsf{L}f^*B)$ is in $\mathsf{D}^{\mathsf{b}}_{\mathsf{c}}(X)$.

The desired assertions now result from the isomorphisms

$$\begin{aligned}
\mathsf{R}\mathcal{H}om_X(\mathsf{R}\mathcal{H}om_X(\mathsf{L}f^*M, \mathsf{L}f^*B), \mathsf{L}f^*B) &\xrightarrow{\sim} \mathsf{R}\mathcal{H}om_X(\mathsf{L}f^*\mathsf{R}\mathcal{H}om_X(M,B), \mathsf{L}f^*B)\\
&\xrightarrow{\sim} \mathsf{R}\mathcal{H}om_X(f^!\mathsf{R}\mathcal{H}om_X(M,B), f^!B)\\
&\xrightarrow{\sim} \mathsf{R}\mathcal{H}om_X(\mathsf{R}\mathcal{H}om_X(\mathsf{L}f^*M, f^!B), f^!B),
\end{aligned}$$

given by formula (2.1.7.1), Lemma 2.1.10, and formula (2.1.7.2), respectively. □

2.2. **Ascent and descent.** Let $f\colon X \to Y$ be a scheme-map.

Remark 2.2.1. Recall that f is said to be *faithfully flat* if it is flat and surjective; and that for any flat f, the canonical map to f^* from its left-derived functor $\mathsf{L}f^*$ is an isomorphism—in brief, $\mathsf{L}f^* = f^*$.

Lemma 2.2.2. *Let $f\colon X \to Y$ be a perfect scheme-map and M a complex in $\mathsf{D}(Y)$.*

If M is in $\mathsf{D}^{\mathsf{b}}_{\mathsf{c}}(Y)$ then $\mathsf{L}f^*M$ is in $\mathsf{D}^{\mathsf{b}}_{\mathsf{c}}(X)$. The converse holds when M is in $\mathsf{D}_{\mathsf{c}}(Y)$ and f is faithfully flat, or proper and surjective.

Proof. The forward implication is contained in Remark 2.1.4.

For the converse, when f is faithfully flat there are isomorphisms $H^n(f^*M) \cong f^*H^n(M)$ $(n \in \mathbb{Z})$; so it suffices that $f^*H^n(M) = 0$ imply $H^n(M) = 0$. This can be seen stalkwise, where we need only recall, for a flat local homomorphism $R \to S$ of local rings and any R-module P, that $P \otimes_R S = 0$ implies $P = 0$.

When f is proper then by Remark 2.1.8, $\mathsf{R}f_*(\mathsf{D}^{\mathsf{b}}_{\mathsf{c}}(X)) \subseteq \mathsf{D}^{\mathsf{b}}_{\mathsf{c}}(Y)$ and $\mathsf{R}f_*\mathcal{O}_X$ is perfect. Furthermore, surjectivity of f implies that

$$\operatorname{Supp}_Y \mathsf{R}f_*\mathcal{O}_X \supseteq \operatorname{Supp}_Y H^0\mathsf{R}f_*\mathcal{O}_X = \operatorname{Supp}_Y f_*\mathcal{O}_X = Y.$$

In view of the projection isomorphism

$$\mathsf{R}f_*\mathsf{L}f^*M \simeq \mathsf{R}f_*\mathcal{O}_X \otimes^{\mathsf{L}}_Y M\,,$$

see (B.1.4), the desired converse follows from Theorem 1.4.5. □

Proposition 2.2.3. *Let $f\colon X \to Y$ be a scheme-map and $M \in \mathsf{D}^{-}_{\mathsf{c}}(Y)$.*

If M is perfect, then $\mathsf{L}f^*M$ is perfect. The converse holds if f is faithfully flat, or if f is perfect, proper and surjective.

Proof. Suppose M is perfect in $\mathsf{D}(Y)$. One may assume, after passing to a suitable open cover, that M is a bounded complex of finite-rank free $\mathcal{O}_Y$-modules. Then $\mathsf{L}f^*M = f^*M$ is a bounded complex of finite-rank free $\mathcal{O}_X$-modules. Thus if M is perfect then so is $\mathsf{L}f^*M$.

For the *converse,* when f is faithfully flat we use the following characterization of perfection ([16, p. 135, 5.8.1]): *$M \in \mathsf{D}(Y)$ is perfect if and only if $M \in \mathsf{D}^{\mathsf{b}}_{\mathsf{c}}(Y)$ and there are integers $m \le n$ such that for all $\mathcal{O}_Y$-modules E and all i outside the interval $[m,n]$, $H^i(E \otimes^{\mathsf{L}}_Y M) = 0$.*

Writing f^* in place of $\mathsf{L}f^*$ (see Remark 2.2.1) we have, as in the proof of Lemma 2.2.2, that for any i, the vanishing of

$$H^i(f^*E \otimes^{\mathsf{L}}_X f^*M) = H^i(f^*(E \otimes^{\mathsf{L}}_Y M)) \cong f^*H^i(E \otimes^{\mathsf{L}}_Y M)$$

implies that of $H^i(E \otimes^{\mathsf{L}}_Y M)$. Hence the converse holds.

When f is perfect, proper and surjective, one can argue as in the last part of the proof of Lemma 2.2.2 to show that if $\mathsf{L}f^*M$ is perfect then M is perfect. □

Proposition 2.2.4. *Let $f\colon X \to Y$ be a proper scheme-map and $B \in \mathsf{D}^{+}_{\mathsf{qc}}(Y)$.*

If $F \in \mathsf{D}(X)$ is derived $f^!B$-reflexive then $\mathsf{R}f_*F$ is derived B-reflexive.

Proof. Since F and $\mathsf{R}\mathcal{H}om_X(F, f^!B)$ are in $\mathsf{D}^{\mathsf{b}}_{\mathsf{c}}(X)$, it follows from Remark 2.1.8 that $\mathsf{R}f_*F$ is in $\mathsf{D}^{\mathsf{b}}_{\mathsf{c}}(Y)$, and (via (B.6.1)) that

$$\mathsf{R}\mathcal{H}om_Y(\mathsf{R}f_*F, B) \simeq \mathsf{R}f_*\mathsf{R}\mathcal{H}om_X(F, f^!B) \in \mathsf{D}^{\mathsf{b}}_{\mathsf{c}}(Y).$$

Now apply the functor $\mathsf{R}f_*$ to the assumed isomorphism

$$\delta^{f^!B}_F\colon F \xrightarrow{\ \sim\ } \mathsf{R}\mathcal{H}om_X(\mathsf{R}\mathcal{H}om_X(F, f^!B), f^!B),$$

and use the duality isomorphism (B.6.1) twice, to get the isomorphisms

$$\begin{aligned}\mathsf{R}f_*F &\xrightarrow{\ \sim\ } \mathsf{R}f_*\mathsf{R}\mathcal{H}om_X(\mathsf{R}\mathcal{H}om_X(F, f^!B), f^!B)\\ &\xrightarrow{\ \sim\ } \mathsf{R}\mathcal{H}om_Y(\mathsf{R}f_*\mathsf{R}\mathcal{H}om_X(F, f^!B), B)\\ &\xrightarrow{\ \sim\ } \mathsf{R}\mathcal{H}om_Y(\mathsf{R}\mathcal{H}om_Y(\mathsf{R}f_*F, B), B).\end{aligned}$$

Their composition is actually $\delta^B_{\mathsf{R}f_*F}$, though that doesn't seem so easy to show. Fortunately, owing to Proposition 1.3.3(ii) we needn't do so to conclude that $\mathsf{R}f_*F$ is derived B-reflexive. □

Theorem 2.2.5. *Let $f\colon X \to Y$ be a perfect scheme-map, M a complex in $\mathsf{D}^-_{\mathsf{c}}(Y)$, and B a complex in $\mathsf{D}^+_{\mathsf{c}}(Y)$.*

If M is derived B-reflexive, then $\mathsf{L}f^*M$ is derived $\mathsf{L}f^*B$-reflexive and derived $f^!B$-reflexive. Conversely, if f is faithfully flat, or proper and surjective, and $\mathsf{L}f^*M$ is derived $\mathsf{L}f^*B$-reflexive or derived $f^!B$-reflexive, then M is derived B-reflexive.

Proof. Suppose first that M is derived B-reflexive, so that, by definition, both M and $\mathsf{R}\mathcal{H}om_Y(M,B)$ are in $\mathsf{D}^{\mathsf{b}}_{\mathsf{c}}(Y)$. Then (2.1.7.1) and Remark 2.1.4 show that $\mathsf{L}f^*M$ and $\mathsf{R}\mathcal{H}om_X(\mathsf{L}f^*M, \mathsf{L}f^*B)$ are in $\mathsf{D}^{\mathsf{b}}_{\mathsf{c}}(X)$. Moreover, application of the functor $\mathsf{L}f^*$ to the $\mathsf{D}(Y)$-isomorphism $M \simeq \mathsf{R}\mathcal{H}om_Y(\mathsf{R}\mathcal{H}om_Y(M,B),B)$ yields

$$\mathsf{L}f^*M \simeq \mathsf{R}\mathcal{H}om_X(\mathsf{R}\mathcal{H}om_X(\mathsf{L}f^*M, \mathsf{L}f^*B), \mathsf{L}f^*B)$$

in $\mathsf{D}(X)$. This implies that $\mathsf{L}f^*M$ is derived $\mathsf{L}f^*B$-reflexive; see Proposition 1.3.3(ii). When B is in $\mathsf{D}^+_{\mathsf{c}}(Y)$, Lemma 2.1.11 yields that $\mathsf{L}f^*M$ is derived $f^!B$-reflexive.

Suppose, conversely, that $\mathsf{L}f^*M$ is derived $\mathsf{L}f^*B$-reflexive, or equivalently, that $\mathsf{L}f^*M$ is derived $f^!B$-reflexive (see Lemma 2.1.11). Then, first of all, $\mathsf{L}f^*M \in \mathsf{D}^{\mathsf{b}}_{\mathsf{c}}(X)$ and, by (2.1.7.1), $\mathsf{L}f^*\mathsf{R}\mathcal{H}om_Y(M,B) \in \mathsf{D}^{\mathsf{b}}_{\mathsf{c}}(X)$. Lemma 2.2.2 gives then that $M \in \mathsf{D}^{\mathsf{b}}_{\mathsf{c}}(Y)$, and similarly, since $\mathsf{R}\mathcal{H}om_Y(M,B) \in \mathsf{D}_{\mathsf{c}}(Y)$ (see [15, p. 92, 3.3]), that $\mathsf{R}\mathcal{H}om_Y(M,B) \in \mathsf{D}^{\mathsf{b}}_{\mathsf{c}}(Y)$.

Next, when f is faithfully flat (so that $\mathsf{L}f^* = f^*$, see Remark 2.2.1), one checks, with moderate effort, that if

$$\delta := \delta^B_M \colon M \to \mathsf{R}\mathcal{H}om_Y(\mathsf{R}\mathcal{H}om_Y(M,B),B)$$

is the canonical $\mathsf{D}(Y)$-map, then $f^*\delta$ is identified, via (2.1.7.1), with the canonical $\mathsf{D}(X)$-map $\delta^{f^*B}_{f^*M}$. The latter being an isomorphism, therefore so are all the maps $H^n(f^*\delta) = f^*H^n(\delta)$. Verifying that a sheaf-map is an isomorphism can be done stalkwise, and so, f being faithfully flat, local considerations show that the maps $H^n(\delta)$ are isomorphisms. Therefore, δ is an isomorphism.

Finally, when f is proper and surjective and $\mathsf{L}f^*M$ is derived $f^!B$-reflexive, whence, by Proposition 2.2.4, $\mathsf{R}f_*M$ is derived B-reflexive, one can argue as in the last part of the proof of Lemma 2.2.2 to deduce that M is derived B-reflexive. □

Taking $M = \mathcal{O}_Y$ one gets:

Corollary 2.2.6. *Let $f\colon X \to Y$ be a perfect scheme-map and $B \in \mathsf{D}^+_{\mathsf{c}}(Y)$.*

If B is semidualizing, then so are $\mathsf{L}f^*B$ and $f^!B$. Conversely, if f is faithfully flat, or proper and surjective, and $\mathsf{L}f^*B$ or $f^!B$ is semidualizing, then so is B. □

Corollary 2.2.7. *Let $f\colon X \to Y$ be a perfect scheme-map and M a complex in $\mathsf{D}^-_{\mathsf{c}}(Y)$. Consider the following properties:*

(a) *M is semidualizing.*
(b) *M is derived $\mathcal{O}_Y$-reflexive.*
(c) *M is invertible.*

Each of these properties implies the corresponding property for $\mathsf{L}f^*M$ in $\mathsf{D}(X)$. The converse holds when f is faithfully flat, or proper and surjective.

Proof. Note that, given Lemma 2.2.2, we may assume that M is in $\mathsf{D}^{\mathsf{b}}_{\mathsf{c}}(Y)$. The assertions about properties (a) and (b) are the special cases $(M,B) = (\mathcal{O}_Y, M)$ and $(M,B) = (M, \mathcal{O}_Y)$, respectively, of Theorem 2.2.5. The assertion about (c) follows from the assertion about (a) together with Proposition 2.2.3. □

2.3. **Gorenstein-perfect maps.** Let $f\colon X \to Y$ be a scheme-map.

Definition 2.3.1. A *relative dualizing complex* for f is any $\mathcal{O}_X$-complex isomorphic in $\mathsf{D}(X)$ to $f^!\mathcal{O}_Y$.

Any relative dualizing complex is in $\mathsf{D}^+_{\mathsf{c}}(X)$. Indeed, §§B.3(i) and B.4 reduce the assertion to the case of maps between affine schemes, where the desired assertion follows from the following example.

Example 2.3.2. For a homomorphism $\tau\colon K \to P$ of commutative rings we write Ω_τ for the P-module of relative differentials, and set

$$\Omega^n_\tau = \textstyle\bigwedge^n_P \Omega_\tau \quad \text{for each } 0 \le n \in \mathbb{Z}.$$

Let $\sigma\colon K \to S$ be a homomorphism of rings that is essentially of finite type; thus, there exists a factorization

$$K \xrightarrow{\dot\sigma} P \xrightarrow{\sigma'} S \tag{2.3.2.1}$$

where $\dot\sigma$ is *essentially smooth of relative dimension* d and σ' is *finite*, see (A.1). As in [5, (8.0.2)], we set

$$D^\sigma := \Sigma^d \,\mathsf{R}\mathrm{Hom}_P(S, \Omega^d_{\dot\sigma}) \in \mathsf{D}(S). \tag{2.3.2.2}$$

With $f\colon X = \operatorname{Spec} S \to \operatorname{Spec} K = Y$ the scheme-map corresponding to σ, the complex of $\mathcal{O}_X$-modules $(D^\sigma)^\sim$ is a relative dualizing complex for f; in particular, up to isomorphism, D^σ *depends only on* σ, and not on the factorization (2.3.2.1).

Indeed, there is a $\mathsf{D}_{\mathsf{qc}}(X)$-isomorphism

$$f^!\mathcal{O}_Y \simeq (D^\sigma)^\sim; \tag{2.3.2.3}$$

for, if $f = \dot f f'$ is the factorization corresponding to (2.3.2.1) then

$$f^!\mathcal{O}_Y \simeq f'^!\dot f^!\mathcal{O}_Y \simeq f'^!(\Sigma^d\Omega^d_{\dot\sigma})^\sim \simeq \Sigma^d \,\mathsf{R}\mathrm{Hom}_P(S, \Omega^d_{\dot\sigma})^\sim = (D^\sigma)^\sim,$$

the second isomorphism coming from §B.5, and the third from (B.6.2).

Definition 2.3.3. A complex F in $\mathsf{D}(X)$ is said to be *G-perfect* (for *Gorenstein-perfect*) *relative to* f if F is derived $f^!\mathcal{O}_Y$-reflexive. The full subcategory of $\mathsf{D}^{\mathsf{b}}_{\mathsf{c}}(X)$, whose objects are the complexes that are G-perfect relative to f is denoted $\mathsf{G}(f)$.

In particular, F is in $\mathsf{G}(\mathrm{id}^X)$ if and only if F is derived $\mathcal{O}_X$-reflexive. We set

$$\mathsf{G}(X) := \mathsf{G}(\mathrm{id}^X)\,.$$

In view of (2.3.2.3), in the affine case G-perfection can be expressed in terms of *finite G-dimension* in the sense of Auslander and Bridger [1]; see [5, §6.3 and 8.2.1].

As is the case for perfection (Remark 2.1.2), G-perfection can be tested locally.

Remark 2.3.4. A complex F in $\mathsf{D}(X)$ is in $\mathsf{G}(f)$ if and only if every $x \in Z$ has an open neighborhood U such that $F|_U$ is in $\mathsf{G}(f|_U)$.

If f factors as $X \xrightarrow{i} W \xrightarrow{h} Y$ with i a closed immersion and h essentially smooth, then F is in $\mathsf{G}(f)$ if and only if i_*F is in $\mathsf{G}(W)$. It suffices to show this locally; and then this is [5, 8.2.1], in view of the equivalence of categories in Example 1.1.3.

Definition 2.3.5. The map $f\colon X \to Y$ is said to be *G-perfect* (for *Gorenstein perfect*) if $f^!\mathcal{O}_Y$ is semidualizing, that is, if $\mathcal{O}_X$ is in $\mathsf{G}(f)$.

A local theory of such maps already exists:

Example 2.3.6. If $X = \operatorname{Spec} S$ and $Y = \operatorname{Spec} K$, where K and S are noetherian rings, and $\sigma\colon K \to S$ is the ring-homomorphism corresponding to f, then f is *G-perfect* if and only if σ is of *finite G-dimension* in the sense of [3]; see [5, 8.4.1].

Recall from Remark 2.1.5 that f is perfect if and only if $f^!\mathcal{O}_Y$ is in $\mathsf{P}(f)$, the full subcategory of $\mathsf{D}(X)$ whose objects are all the complexes that are perfect with respect to f. There is a similar description of G-perfection:

Remark 2.3.7. The map f is G-perfect if and only if $f^!\mathcal{O}_Y \in \mathsf{G}(f)$. This follows from Proposition 1.3.3, since for all $x \in X$, the stalk $(f^!\mathcal{O}_Y)_x \not\simeq 0$; see (2.3.2.3).

Remark 2.3.8. When Y is Gorenstein, *every* map $f\colon X \to Y$ is G-perfect: [5, 8.3.1] and (2.3.2.3) together imply that $\mathsf{G}(f) = \mathsf{D}^{\mathsf{b}}_{\mathsf{c}}(X)$.

Via (2.3.2.3), a slight generalization of [16, p. 258, 4.9*ff*] globalizes [6, 1.2]:

Proposition 2.3.9. *Let $f\colon X \to Y$ be a scheme-map.*

The following inclusion holds: $\mathsf{P}(f) \subseteq \mathsf{G}(f)$.

If $M \in \mathsf{P}(Y)$ then the functor $\mathsf{R}\mathcal{H}om_X(-, f^!M)$ takes $\mathsf{P}(f)$ (resp. $\mathsf{G}(f)$) into itself; and if $M \in \mathsf{G}(Y)$ then $\mathsf{R}\mathcal{H}om_X(-, f^!M)$ takes $\mathsf{P}(f)$ into $\mathsf{G}(f)$.

Proof. The first assertion is a restatement of Remark 2.1.6.

The second assertion is local on X, so one may suppose f factors as $X \xrightarrow{i} W \xrightarrow{h} Y$ with i a closed immersion and h essentially smooth. For any $F \in \mathsf{D}^{\mathsf{b}}_{\mathsf{c}}(X)$ and $M \in \mathsf{D}^{+}_{\mathsf{qc}}(Y)$ one has, using formula (B.6.1), §B.5 and Lemma 1.4.6.

$$i_*\mathsf{R}\mathcal{H}om_X(F, i^!h^!M) \simeq \mathsf{R}\mathcal{H}om_W(i_*F, h^!M) \simeq \mathsf{R}\mathcal{H}om_W(i_*F, h^*M) \otimes^{\mathsf{L}}_W h^!\mathcal{O}_Y,$$

where $h^!\mathcal{O}_Y$ is invertible. Consequently, by Remark 2.1.2,

$$\begin{aligned}\mathsf{R}\mathcal{H}om_X(F, f^!M) \in \mathsf{P}(f) &\iff i_*\mathsf{R}\mathcal{H}om_X(F, f^!M) \in \mathsf{P}(W)\\ &\iff \mathsf{R}\mathcal{H}om_W(i_*F, h^*M) \in \mathsf{P}(W).\end{aligned}$$

Similarly, by Remark 2.3.4 and Corollary 1.5.4(2),

$$\begin{aligned}\mathsf{R}\mathcal{H}om_X(F, f^!M) \in \mathsf{G}(f) &\iff i_*\mathsf{R}\mathcal{H}om_X(F, f^!M) \in \mathsf{G}(h)\\ &\iff \mathsf{R}\mathcal{H}om_W(i_*F, h^*M) \text{ is derived } \mathcal{O}_W\text{-reflexive.}\end{aligned}$$

If $F \in \mathsf{P}(f)$ then i_*F is a perfect $\mathcal{O}_W$-complex, and by Lemma 1.4.6(2),

$$\mathsf{R}\mathcal{H}om_W(i_*F, h^*M) \simeq h^*M \otimes^{\mathsf{L}}_W \mathsf{R}\mathcal{H}om_W(i_*F, \mathcal{O}_W), \tag{2.3.9.1}$$

where $\mathsf{R}\mathcal{H}om_W(i_*F, \mathcal{O}_W)$ is perfect (see Theorem 1.4.3).

If $M \in \mathsf{P}(Y)$ then by Proposition 2.2.3, $h^*M \in \mathsf{P}(W)$, and then (2.3.9.1) shows that $\mathsf{R}\mathcal{H}om_W(i_*F, h^*M) \in \mathsf{P}(W)$. Thus $\mathsf{R}\mathcal{H}om_X(F, f^!M) \in \mathsf{P}(f)$.

If $M \in \mathsf{G}(Y)$, then h^*M is derived $\mathcal{O}_W$-reflexive, hence so is $\mathsf{R}\mathcal{H}om_W(i_*F, h^*M)$; see Theorem 2.2.5, (2.3.9.1) and Proposition 1.4.4. So $\mathsf{R}\mathcal{H}om_X(F, f^!M) \in \mathsf{G}(f)$.

If $F \in \mathsf{G}(f)$ and $M \in \mathsf{P}(Y)$ then $i_*F \in \mathsf{G}(h)$ is $\mathcal{O}_W$-reflexive and h^*M is perfect; so by Lemma 1.4.6(1), (2.3.9.1) still holds, so $\mathsf{R}\mathcal{H}om_W(i_*F, h^*M)$ is $\mathcal{O}_W$-reflexive, by Remark 1.3.4 and Proposition 1.4.4. So again, $\mathsf{R}\mathcal{H}om_X(F, f^!M) \in \mathsf{G}(f)$. □

From Proposition 2.3.9 one gets the following result. It can also be seen as the special case $g = \mathrm{id}^Y$ of Proposition 2.5.2 below.

Corollary 2.3.10. *Any perfect map is G-perfect.* □

Applying Proposition 2.3.9 to $\mathsf{R}\mathcal{H}om_X(\mathcal{O}_X, f^!F)$, one gets:

Corollary 2.3.11. *If $f\colon X \to Y$ is perfect then $f^!\mathsf{P}(Y) \subseteq \mathsf{P}(f)$ and $f^!\mathsf{G}(Y) \subseteq \mathsf{G}(f)$. If f is G-perfect then $f^!\mathsf{P}(Y) \subseteq \mathsf{G}(f)$.* □

Also, in view of Proposition 1.3.3(iii):

Corollary 2.3.12. *For any scheme-map $f\colon X \to Y$, the* relative dualizing functor $\mathsf{R}\mathcal{H}om_X(-, f^!\mathcal{O}_Y)$ *induces a commutative diagram of categories, where horizontal arrows represent equivalences:*

$$\begin{array}{ccc} \mathsf{G}(f)^{\mathsf{op}} & \xleftrightarrow{\ \equiv\ } & \mathsf{G}(f) \\ \cup\!\mid & & \cup\!\mid \\ \mathsf{P}(f)^{\mathsf{op}} & \xleftrightarrow{\ \equiv\ } & \mathsf{P}(f) \end{array}$$

These equivalences are dualities, *in the sense of* [5, §6]. □

2.4. **Quasi-Gorenstein maps.** For the following notion of quasi-Gorenstein map, cf. [4, 2.2] and [5, §8.6.1]. For the case when f is flat, see also [15, p. 298, Exercise 9.7], which can be done, e.g., along the lines of the proof of [18, Lemma 1].)

Definition 2.4.1. A map $f\colon X \to Y$ is *quasi-Gorenstein* if $f^!\mathcal{O}_Y$ is invertible. If, in addition, f is perfect, then f is said to be a *Gorenstein map.*

If $f\colon X \to Y$ is quasi-Gorenstein then, clearly, $\mathcal{O}_X \in \mathsf{G}(f)$, i.e., f *is G-perfect.* More generally, Corollary 1.5.4 shows that $\mathsf{G}(f) = \mathsf{G}(X)$.

Example 2.4.2. Let $f\colon X \to Y$ be a scheme map. If X is Gorenstein and f is G-perfect, then f is quasi-Gorenstein; see Example 1.5.5. Example 2.3.8 shows then that when X and Y are both Gorenstein f is quasi-Gorenstein.

One has the following globalization of the *flat* case of [5, 8.6.2], see also [4, 2.4]:

Proposition 2.4.3. *If $f\colon X \to Y$ is a flat Gorenstein map, with diagonal map $\delta\colon X \to X \times_Y X$, then there are natural isomorphisms*

$$W_f := \mathcal{H}om_X(\delta^!(\mathcal{O}_{X\times_Y X}), \mathcal{O}_X) \underset{\nu}{\xrightarrow{\ \sim\ }} \mathsf{R}\mathcal{H}om_X(\delta^!(\mathcal{O}_{X\times_Y X}), \mathcal{O}_X) \xrightarrow{\ \sim\ } f^!\mathcal{O}_Y\,.$$

If furthermore $g\colon Z \to X$ is finite, then (B.6.1) *gives a natural isomorphism*

$$g_*(fg)^!\mathcal{O}_Y \cong \mathsf{R}g_*g^!f^!\mathcal{O}_Y \xrightarrow{\ \sim\ } \mathsf{R}\mathcal{H}om_X(g_*\mathcal{O}_Z, W_f)\,.$$

Proof. For *any* flat scheme-map $f\colon X \to Y$ there is a natural isomorphism

$$\delta^!(\mathcal{O}_{X\times_Y X}) \xrightarrow{\ \sim\ } \mathsf{R}\mathcal{H}om_X(f^!\mathcal{O}_Y, \mathcal{O}_X)$$

(see Corollary 6.5 in [6], with $M = \mathcal{O}_X = N$).

It follows, when $f^!\mathcal{O}_Y$ is invertible, that the complex $\delta^!(\mathcal{O}_{X\times_Y X})$ is invertible, and that there is a natural $\mathsf{D}(X)$-isomorphism

$$f^!\mathcal{O}_Y \xrightarrow{\ \sim\ } \mathsf{R}\mathcal{H}om_X(\delta^!(\mathcal{O}_{X\times_Y X}), \mathcal{O}_X).$$

That the natural map ν is an isomorphism holds true with any perfect complex in place of $\delta^!(\mathcal{O}_{X\times_Y X})$: the assertion is local, hence reduces to the corresponding (obvious) assertion for rings.

For the final assertion, note that the natural map is an isomorphism

$$g_*(fg)^!\mathcal{O}_Y \xrightarrow{\ \sim\ } \mathsf{R}g_*(fg)^!\mathcal{O}_Y$$

because the equivalence of categories given in [15, p. 133, 7.19] allows us to work exclusively with quasi-coherent sheaves, on which the functor g_* is exact. □

2.5. **Composition, decomposition, and base change.** We turn now to the behavior of relative perfection and G-perfection, especially vis-à-vis the derived direct- and inverse-image functors and the twisted inverse image functor, when several maps are involved.

Generalizing Proposition 2.2.3 (which is the special case $f = \mathrm{id}^X$), one has:

Proposition 2.5.1 (cf. [16, pp. 253–254, 4.5.1])**.** *Let $Z \xrightarrow{g} X \xrightarrow{f} Y$ be scheme-maps, with g perfect.*

Then $\mathsf{L}g^*\mathsf{P}(f) \subseteq \mathsf{P}(fg)$. In particular, if f is perfect then so is fg.

Conversely, if g is faithfully flat, or if g is proper and surjective and $F \in \mathsf{D_c}(X)$, then $\mathsf{L}g^*F \in \mathsf{P}(fg) \implies F \in \mathsf{P}(f)$. In particular, if fg is perfect then so is f.

Proof. Let $F \in \mathsf{P}(f)$. By Lemma 2.2.2, $\mathsf{L}g^*F \in \mathsf{D}^{\mathsf{b}}_{\mathsf{c}}(Z)$. Hence by [16, p. 242, 3.3, p. 251, 4.3 and p. 115, 3.5(b)] (whose proofs are easily made to apply to essentially finite-type maps of noetherian schemes), for $\mathsf{L}g^*F$ to be in $\mathsf{P}(fg)$ it suffices that there be integers $m \le n$ such that for any $\mathcal{O}_Y$-module M and integer $j \notin [m,n]$,

$$0 = H^j(\mathsf{L}g^*F \otimes^{\mathsf{L}}_Z \mathsf{L}(fg)^*M) \cong H^j(\mathsf{L}g^*(F \otimes^{\mathsf{L}}_X \mathsf{L}f^*M)).$$

But by *loc. cit.* this holds because F is in $\mathsf{P}(g)$ and $\mathsf{L}g^*$ is bounded.

Taking $M = \mathcal{O}_Y$ one gets that if f is perfect then fg is perfect.

For the converse, if g is faithfully flat (so that $\mathsf{L}g^* = g^*$) then for any $\mathcal{O}_X$-module F and any $j \in \mathbb{Z}$, one sees stalkwise that

$$H^j(g^*F) \cong g^*H^j(F) = 0 \iff H^j(F) = 0.$$

Hence if $F \in \mathsf{D_c}(X)$ and $g^*F \in \mathsf{P}(fg) \subseteq \mathsf{D}^{\mathsf{b}}_{\mathsf{c}}(Z)$—whence $F \in \mathsf{D}^{\mathsf{b}}_{\mathsf{c}}(X)$—then by an argument like that above, $F \in \mathsf{P}(f)$.

In the remaining case one argues as in the proof of Proposition 2.2.3. (It should be noted that the relevant part of Theorem 1.4.5 is proved via the above criterion for relative perfection, so it applies not only to perfection but more generally to relative perfection.) □

Analogously, for $A := f^!\mathcal{O}_Y$ one has $(fg)^!\mathcal{O}_Y \simeq g^!A$, so Theorem 2.2.5 gives

Proposition 2.5.2 (Cf. [3, 4.7])**.** *Let $Z \xrightarrow{g} X \xrightarrow{f} Y$ be scheme-maps, with g perfect.*

Then $\mathsf{L}g^*\mathsf{G}(f) \subseteq \mathsf{G}(fg)$. In particular, if f is G-perfect then so is fg.

Conversely, if g is faithfully flat and $F \in \mathsf{D}^-_{\mathsf{c}}(X)$, or if g is proper and surjective and $F \in \mathsf{D_c}(X)$, then $\mathsf{L}g^*F$ in $\mathsf{G}(fg)$ implies $F \in \mathsf{G}(f)$. □

The next proposition generalizes parts of Proposition 2.3.9. The proof is quite similar, and so is omitted.

Proposition 2.5.3. *Let $Z \xrightarrow{g} X \xrightarrow{f} Y$ be scheme-maps, $P \in \mathsf{P}(g)$, $F, A \in \mathsf{D}(X)$.*

If $F \in \mathsf{P}(f)$ then $\mathsf{R}\mathcal{H}om_Z(P, g^!F) \in \mathsf{P}(fg)$. (Cf. [16, p. 258, 4.9].) *In other words, the functor $\mathsf{R}\mathcal{H}om_Z(-, g^!F)$ takes $\mathsf{P}(g)$ to $\mathsf{P}(fg)$.*

If F is A-reflexive then $\mathsf{R}\mathcal{H}om_Z(P, g^!F)$ is $g^!A$-reflexive. For $A = f^!\mathcal{O}_Y$ this gives that $\mathsf{R}\mathcal{H}om_X(-, g^!F)$ takes $\mathsf{P}(g)$ to $\mathsf{G}(fg)$. □

Proposition 2.5.4. *Let $Z \xrightarrow{g} X \xrightarrow{f} Y$ be scheme-maps, with g perfect.*

Then $g^!\mathsf{P}(f) \subseteq \mathsf{P}(fg)$ and $g^!\mathsf{G}(f) \subseteq \mathsf{G}(fg)$.

Conversely, if g is proper and surjective, F is in $\mathsf{D}^+_{\mathsf{c}}(X)$, and $g^!F$ is in $\mathsf{P}(fg)$ (resp. $\mathsf{G}(fg)$) then F is in $\mathsf{P}(f)$ (resp. $\mathsf{G}(f)$).

Proof. The direct assertions are obtained from Proposition 2.5.3 by taking $P = \mathcal{O}_Z$.
If g is perfect then $g^!\mathcal{O}_X \in \mathsf{P}(g)$ and

$$\mathsf{R}g_*g^!F \simeq \mathsf{R}g_*(g^!\mathcal{O}_X \otimes^{\mathsf{L}}_Z \mathsf{L}g^*F) \simeq \mathsf{R}g_*g^!\mathcal{O}_X \otimes^{\mathsf{L}}_X F\,;$$

see Remark 2.1.5. If g is also proper then $\mathsf{R}g_*g^!\mathcal{O}_X$ is perfect [16, p. 257, 4.8(a)]. One can then argue as at the end of the proof of Proposition 2.5.1. □

Proposition 2.5.5. *Let $Z \xrightarrow{g} X \xrightarrow{f} Y$ be scheme-maps, with g proper.*
Then $\mathsf{R}g_*\mathsf{P}(fg) \subseteq \mathsf{P}(f)$ and $\mathsf{R}g_*\mathsf{G}(fg) \subseteq \mathsf{G}(f)$.

Proof. For P one can proceed as in [16, p. 257, 4.8]. (This ultimately uses the projection isomorphism (B.1.3).)
For G apply Proposition 2.2.4 with $B = f^!\mathcal{O}_Y$. □

Proposition 2.5.6 (Cf. [17, 5.2])**.** *Let $Z \xrightarrow{g} X \xrightarrow{f} Y$ be scheme-maps, with f quasi-Gorenstein.*
Then $\mathsf{G}(fg) = \mathsf{G}(g)$. In particular, fg is G-perfect if and only if so is g.
Also, if g is quasi-Gorenstein then so is fg.

Proof. For any invertible $F \in \mathsf{D}(X)$ the natural map (see 2.1.5.1)

$$g^!\mathcal{O}_X \otimes^{\mathsf{L}}_Z \mathsf{L}g^*F \to g^!F$$

is an *isomorphism*: the question being local (see §B.4), one reduces via 1.5.2(iii′) to the simple case $F = \mathcal{O}_X$.
When F is the invertible complex $f^!\mathcal{O}_Y$, there results an isomorphism

$$g^!\mathcal{O}_X \otimes^{\mathsf{L}}_Z \mathsf{L}g^*f^!\mathcal{O}_Y \to g^!f^!\mathcal{O}_Y \simeq (fg)^!\mathcal{O}_Y.$$

The first assertion follows from Corollary 1.5.4(1) (with $A = g^!\mathcal{O}_X$, $L = \mathsf{L}g^*f^!\mathcal{O}_Y$); and the last holds because if $g^!\mathcal{O}_X$ is invertible then by Corollary 1.5.3, $(fg)^!\mathcal{O}_Y$ is invertible as well. □

The last assertion of Proposition 2.5.6 expresses a composition property of quasi-Gorenstein homomorphisms. Here is a decomposition property:

Proposition 2.5.7 (Cf. [2, 4.6], [17, 5.5])**.** *Let $Z \xrightarrow{g} X \xrightarrow{f} Y$ be scheme-maps, with g perfect.*
If fg is quasi-Gorenstein then g is Gorenstein.
Suppose g is faithfully flat, or proper and surjective. If fg is quasi-Gorenstein (resp. Gorenstein) then so is f.

Proof. By Remark 2.1.5, one has $g^!\mathcal{O}_X \in \mathsf{D}^{\mathsf{b}}_{\mathsf{c}}(Z)$ and an isomorphism

$$g^!\mathcal{O}_X \otimes^{\mathsf{L}}_Z \mathsf{L}g^*f^!\mathcal{O}_Y \to g^!f^!\mathcal{O}_Y \simeq (fg)^!\mathcal{O}_Y.$$

Also, the paragraph immediately before §5.5 in [21] yields $f^!\mathcal{O}_Y \in \mathsf{D}_{\mathsf{c}}(X)$, whence $\mathsf{L}g^*f^!\mathcal{O}_Y \in \mathsf{D}_{\mathsf{c}}(Z)$. Now Corollary 1.5.3(2) gives the first assertion. It also shows that $\mathsf{L}g^*f^!\mathcal{O}_Y$ is invertible, whence so is $f^!\mathcal{O}_Y$ if g is faithfully flat, or proper and surjective (see Corollary 2.2.7), giving the quasi-Gorenstein part of the second assertion. The last assertion in Proposition 2.5.2 now gives the Gorenstein part. □

From Propositions 2.5.2, 2.5.4 and 2.5.6 one gets:

Corollary 2.5.8. *Let there be given a commutative diagram*

$$\begin{tikzcd} X' \arrow[r, "v"] \arrow[d, "h"'] & X \arrow[d, "f"] \\ Y' \arrow[r, "u"'] & Y \end{tikzcd}$$

with u quasi-Gorenstein and v perfect.

Then $\mathsf{L}v^*\mathsf{G}(f) \subseteq \mathsf{G}(h)$ and $v^!\mathsf{G}(f) \subseteq \mathsf{G}(h)$. Thus, when f is G-perfect so is h. □

It is shown in [16, p. 245, 3.5.2] that relative perfection is preserved under tor-independent base change. Here is an analog (and more) for relative G-perfection.

Proposition 2.5.9. *Let there be given a tor-independent fiber square* (*see* §B.2)

$$\begin{tikzcd} X' \arrow[r, "v"] \arrow[d, "h"'] & X \arrow[d, "f"] \\ Y' \arrow[r, "u"'] & Y \end{tikzcd}$$

If the map u is Gorenstein, or flat, or if u is perfect and f is proper, then $\mathsf{L}v^*\mathsf{G}(f) \subseteq \mathsf{G}(h)$. In particular, if f is G-perfect then so is h.

Conversely, suppose that u is faithfully flat, or that u is perfect, proper, and surjective and f is proper. If $F \in \mathsf{D}^{\mathsf{b}}_{\mathsf{c}}(X)$ and $\mathsf{L}v^*F \in \mathsf{G}(h)$, then $F \in \mathsf{G}(f)$.

Proof. In all cases, u is perfect, whence so is v [16, p. 245, 3.5.2].

If u is Gorenstein, the assertion is contained in Corollary 2.5.8.

By Lemma 2.2.2, if F is $f^!\mathcal{O}_Y$-reflexive then $\mathsf{L}v^*F$ is $\mathsf{L}v^*f^!\mathcal{O}_Y$-reflexive.

If u (hence v) is flat then by §B.4, one has

$$\mathsf{L}v^*f^!\mathcal{O}_Y \cong h^!\mathsf{L}u^*\mathcal{O}_Y = h^!\mathcal{O}_{Y'}. \tag{2.5.9.1}$$

Thus v^*F is $h^!\mathcal{O}_{Y'}$-reflexive, i.e., $v^*F \in \mathsf{G}(h)$.

The case when u is perfect and f is proper is treated similarly through the tor-independent base-change theorem [19, 4.4.3].

For the converse, the assumption is, in view of the isomorphism (2.5.9.1), that $\mathsf{L}v^*F$ is derived $\mathsf{L}v^*f^!\mathcal{O}_Y$-reflexive. Formula (2.1.5.3) gives that $f^!\mathcal{O}_Y \in \mathsf{D}^{+}_{\mathsf{c}}(X)$. So since v satisfies all the same hypotheses as u does, Theorem 2.2.5 yields that F is $f^!\mathcal{O}_Y$-reflexive, as asserted. □

Proposition 2.5.10. *Let there be given a tor-independent fiber square* (*see* B.2)

$$\begin{tikzcd} X' \arrow[r, "v"] \arrow[d, "h"'] & X \arrow[d, "f"] \\ Y' \arrow[r, "u"'] & Y \end{tikzcd}$$

with either u flat, or u perfect and f proper.

If the map f is quasi-Gorenstein (*resp. Gorenstein*) *then so is h.*

The converse holds if u (*hence v*) *is faithfully flat, or if u* (*hence v*) *is perfect, proper and surjective and f is proper.*

Proof. As in the proof of Proposition 2.5.9, one has the isomorphism (2.5.9.1). Hence if $f^!\mathcal{O}_Y$ is invertible then so is $h^!\mathcal{O}_{Y'}$ (see Corollary 1.5.3(3)), whence the first quasi-Gorenstein assertion, whose converse follows from Corollary 2.2.7(c). Also, by [16, p. 245, 3.5.2], if f is perfect then so is h, whence the first Gorenstein assertion, whose converse follows from the preceding converse and Proposition 2.5.1 (since u perfect and h perfect implies $hu = fv$ perfect). ☐

3. Rigidity over schemes

As in previous sections, schemes are assumed to be noetherian, and scheme-maps to be essentially of finite type, and separated.

3.1. **Rigid complexes.** Fix a scheme X and a semidualizing $\mathcal{O}_X$-complex A, and for any $F \in \mathsf{D}(X)$ set

$$F^\dagger := \mathsf{R}\mathcal{H}om_X(F, A).$$

Definition 3.1.1. An *A-rigid pair* (F, ρ) is one where $F \in \mathsf{D}^{\mathsf{b}}_{\mathsf{c}}(X)$ and ρ is a $\mathsf{D}(X)$-isomorphism

$$\rho\colon F \xrightarrow{\sim} \mathsf{R}\mathcal{H}om_X(F^\dagger, F).$$

An $\mathcal{O}_X$-complex F is *A-rigid* if there exists a ρ such that (F, ρ) is an A-rigid pair. Such a ρ is called an *A-rigidifying isomorphism for F*.

A *morphism of A-rigid pairs* $(F, \rho) \to (G, \sigma)$ is a $\mathsf{D}(X)$-map $\phi\colon F \to G$ such that the following diagram, with $\tilde{\phi}\colon \mathsf{R}\mathcal{H}om_X(F^\dagger, F) \to \mathsf{R}\mathcal{H}om_X(G^\dagger, G)$ the map induced by ϕ, commutes:

$$\begin{tikzcd}
F \arrow[r, "\rho"] \arrow[d, "\phi"'] & \mathsf{R}\mathcal{H}om_X(F^\dagger, F) \arrow[d, "\tilde{\phi}"] \\
G \arrow[r, "\sigma"'] & \mathsf{R}\mathcal{H}om_X(G^\dagger, G)
\end{tikzcd}$$

The terminology "rigid" is motivated by the fact, contained in Theorem 3.2.1, that *the only automorphism of an A-rigid pair is the identity.*

Example 3.1.2. If R is a ring, $X = \operatorname{Spec} R$, and $M, C \in \mathsf{D}^{\mathsf{b}}_{\mathsf{f}}(R)$ are such that $\mathsf{R}\mathrm{Hom}_R(M, C) \in \mathsf{D}^{\mathsf{b}}_{\mathsf{f}}(R)$, then by Example (1.1.3), M is C-rigid in the sense of [5, §7] if and only if $M^\sim$ is $C^\sim$-rigid in the present sense.

Since $\mathsf{R}\mathcal{H}om$ commutes with restriction to open subsets, an A-rigid pair restricts over any open $U \subseteq X$ to an $A|_U$-rigid pair. However, rigidity is *not* a local condition: any invertible sheaf F is F-rigid, but $\mathcal{O}_X$ is not F-rigid unless $F \cong \mathcal{O}_X$.

On the other hand, *rigid pairs glue,* in the sense explained in Theorem 4 of the Introduction, and generalized in Theorem 3.2.9 below.

The central result of this section, Theorem 3.1.7, a globalization of [5, 7.2], is that any A-rigid F is isomorphic in $\mathsf{D}(X)$ to i_*i^*A, with i the inclusion into X of some open-and-closed subscheme—necessarily the support of F, see (1.1.2.1); or equivalently, $F \simeq IA$ for some idempotent $\mathcal{O}_X$-ideal I, uniquely determined by F (see Appendix C); or equivalently, F is, in $\mathsf{D}(X)$, a direct summand of A.

Example 3.1.3. The pair (A, ρ^A) with ρ^A the natural composite isomorphism

$$\rho^A\colon A \xrightarrow{\sim} \mathsf{R}\mathcal{H}om_X(\mathcal{O}_X, A) \xrightarrow{\sim} \mathsf{R}\mathcal{H}om_X(\mathsf{R}\mathcal{H}om_X(A, A), A),$$

is A-rigid.

Extending this example a little leads to:

Example 3.1.4. Let $U \subseteq X$ be an open-and-closed subset, and $i\colon U \hookrightarrow X$ the inclusion. Recall that the $\mathcal{O}_U$-module i^*A is semidualizing; see Corollary 2.2.6. If $F \in \mathsf{D}(U)$ is i^*A-rigid then i_*F is A-rigid.

Indeed, if ρ is an i^*A-rigidifying isomorphism for F, then one has isomorphisms

$$\begin{aligned} i_*F &\underset{i_*\rho}{\xrightarrow{\sim}} i_*\mathsf{R}\mathcal{H}om_U\big(\mathsf{R}\mathcal{H}om_U(F, i^*A), F\big) \\ &\xrightarrow{\sim} i_*\mathsf{R}\mathcal{H}om_U\big(i^*\mathsf{R}\mathcal{H}om_X(i_*F, A), F\big) \\ &\xrightarrow{\sim} \mathsf{R}\mathcal{H}om_X\big(\mathsf{R}\mathcal{H}om_X(i_*F, A), i_*F\big), \end{aligned}$$

where the second comes from (B.1.5) (since $i^*i_*F = F$), and the third is a special case of [19, p. 98, (3.2.3.2)] (or see [19, §3.5.4], or just reason directly, using that i_*F vanishes outside U).

The composition of these isomorphisms is A-rigidifying for i_*F.

Definition 3.1.5. The *U-canonical A-rigid pair* $(i_*i^*A, \rho^{i_*i^*A})$ is the one constructed in Example 3.1.4 out of the i^*A-rigid pair (i^*A, ρ^{i^*A}) in Example 3.1.3.

It is well known that *any* monomorphism (resp. epimorphism) in $\mathsf{D}(X)$ is split, i.e., has a left (resp. right) inverse (see e.g., [19, 1.4.2.1]). Thus, when we speak of mono- or epimorphisms, the adjective "split" will usually be omitted.

Lemma 3.1.6. *Let $\theta\colon F \hookrightarrow A$ be a monomorphism in $\mathsf{D}(X)$. Let (A, ρ^A) be the canonical A-rigid pair in Example 3.1.3. There exists a unique A-rigidifying isomorphism ρ for F such that θ is a morphism of rigid pairs $(F, \rho) \to (A, \rho^A)$.*

Proof. It suffices to deal with the situation separately over each connected component of X; so we may assume that X is connected. Then, by Lemma 1.3.7, either $F = 0$ or θ is an isomorphism. In either case the assertion is obvious. □

Theorem 3.1.7. *For any $F \in \mathsf{D}(X)$, the following conditions are equivalent.*

(i) *F is A-rigid.*
(ii) *In $\mathsf{D}(X)$, $F \simeq I \otimes^{\mathsf{L}}_X A \simeq IA \simeq I^\dagger$ for some idempotent $\mathcal{O}_X$-ideal I.*
(iii) *There is an open-and-closed $U \subseteq X$ such that, $i\colon U \hookrightarrow X$ being the inclusion, $F \simeq i_*i^*A$ in $\mathsf{D}(X)$, whence $U = \operatorname{Supp}_X F$.*
(iv) *There is, in $\mathsf{D}(X)$, a monomorphism $F \to A$.*

When they hold, there is a unique ideal I satisfying condition (ii).

Proof. (iii) ⇒ (i). In view of Example 3.1.3, this is contained in Example 3.1.4.

(i) ⇒ (iii). For the last assertion in (iii), since i_*i^*A vanishes outside U, and since for all $x \in U$ one has, in $\mathsf{D}(\mathcal{O}_{U,x})$,

$$0 \neq \mathcal{O}_{U,x} \simeq (\mathsf{R}\mathcal{H}om_U(i^*A, i^*A))_x \simeq \mathsf{R}\mathcal{H}om_{\mathcal{O}_{U,x}}((i_*i^*A)_x, (i_*i^*A)_x)$$

therefore $U = \operatorname{Supp}_X(i_*i^*A)$.

Now let F be A-rigid. Then $U := \operatorname{Supp}_X F$ *is an open-and-closed subset of* X. For, X is covered by open subsets of the form $V = \operatorname{Spec} R$; and with $j\colon V \hookrightarrow X$ the inclusion, the j^*A-rigid complex j^*F (resp. its homology) is the sheafification of $F_V := \mathsf{R}\Gamma(V, F)$ (resp. its homology), so $(\operatorname{Supp}_X F) \cap V = \operatorname{Supp}_R F_V$. But F_V is $\mathsf{R}\Gamma(V, A)$-rigid (since $(F_V)^\sim \cong j^*F$ is j^*A-rigid), so by [5, 7.2], $\operatorname{Supp}_R F_V = U \cap V$ is an open-and-closed subset of V. That U is open-and-closed follows.

Hence, the natural map $F \to i_*i^*F$ is a $\mathsf{D}(X)$-isomorphism; so to prove the theorem we can replace (X, A, F) by (U, i^*A, i^*F), i.e., *we may assume* $\operatorname{Supp}_X F = X$.

In $\mathsf{D}(X)$, the complex $L := F^\dagger$ is isomorphic to H^0L, which is an invertible sheaf: this assertion need only be checked locally, i.e., for affine X, where it is given by [5, 4.9]. (The assumptions of that theorem are satisfied because F and A are both in $\mathsf{D}^{\mathsf{b}}_{\mathsf{c}}(X)$.) The invertible complex L is derived A-reflexive (take $F = \mathcal{O}_X$ in 1.5.4(2)); similarly, so is $L \otimes^{\mathsf{L}}_X L$. Since $\operatorname{Supp}_X A = X$, by Lemma 1.3.7, therefore Proposition 1.3.3(iii) yields that F is derived A-reflexive. So $L^\dagger \simeq F$, and

$$L^\dagger \simeq \mathsf{R}\mathcal{H}om_X(L, L^\dagger) \simeq (L \otimes^{\mathsf{L}}_X L)^\dagger \qquad \text{(see (1.1.1.1))}.$$

Applying the functor † to these isomorphisms we get $L \otimes^{\mathsf{L}}_X L \simeq L$. Tensoring with L^{-1} shows then that $L \simeq \mathcal{O}_X$. Thus $F \simeq L^\dagger \simeq A$.

(iii) $\Rightarrow$ (ii). Associated to any open-and closed $U \subseteq X$ is the unique idempotent $\mathcal{O}_X$-ideal I that is isomorphic to $i_*\mathcal{O}_U$ (Corollary C.3). For this I we have natural isomorphisms, the second from (B.1.3) and the last two from Corollary C.4:

$$i_*i^*A \simeq i_*(\mathcal{O}_U \otimes^{\mathsf{L}}_U i^*A) \simeq i_*\mathcal{O}_U \otimes^{\mathsf{L}}_X A \simeq I \otimes^{\mathsf{L}}_X A \simeq IA \simeq I^\dagger.$$

(ii) $\Rightarrow$ (iii). Given I as in (ii), let $U = \operatorname{Supp}_X I$, with inclusion $i\colon U \hookrightarrow X$, and use the preceding isomorphisms (see Corollary C.3).

(iii) $\Rightarrow$ (iv). If i is as in (iii), then i_*i^*A is a direct summand of A.

(iv) $\Rightarrow$ (i). See Lemma 3.1.6.

It remains to note that the uniqueness of I in (ii) results from

$$\operatorname{Supp}_X IA = \operatorname{Supp}_X(I \otimes^{\mathsf{L}}_X A) = \operatorname{Supp}_X I \cap \operatorname{Supp}_X A = \operatorname{Supp}_X I \cap X = \operatorname{Supp}_X I,$$

see (1.1.2.2). The proof of Theorem 3.1.7 is now completed. $\square$

Define a *direct decomposition* of $F \in \mathsf{D}(X)$ to be a $\mathsf{D}(X)$-isomorphism

$$F \simeq F_1 \oplus F_2 \oplus \cdots \oplus F_n \tag{3.1.7.1}$$

such that no F_i vanishes; call F *indecomposable* if $F \neq 0$ and in any direct decomposition of F, one has $n = 1$. Say that (3.1.7.1) is an *orthogonal decomposition* of F if, in addition, $F_i \otimes^{\mathsf{L}}_X F_j = 0$ for all $i \neq j$.

Corollary 3.1.8. *Let $F \neq 0$ be an A-rigid complex. Let $\operatorname{Supp}_X F = \bigsqcup_{s=1}^n U_s$ be a decomposition into disjoint nonempty connected closed subsets, and $i_s\colon U_s \hookrightarrow X$ $(1 \le s \le n)$ the canonical inclusions.*

The U_s are then connected components of X, and there is an orthogonal decomposition into indecomposable A-rigid complexes: $F \simeq \bigoplus_{s=1}^n (i_s)_*(i_s)^*A$.

If $F \simeq F_1 \oplus \cdots \oplus F_r$ is a direct decomposition with each F_t indecomposable, then $r = n$ and (after renumbering) there is for each s an isomorphism $F_s \simeq (i_s)_*(i_s)^*A$.

Proof. Since by Theorem 3.1.7(iii), $\operatorname{Supp}_X F$ is open and closed in X, therefore each U_s is a connected component of X. Moreover, if $i\colon \operatorname{Supp}_X F \hookrightarrow X$ is the inclusion, then i^*A is semidualizing (Corollary 2.2.6), and compatibility of $\mathsf{R}\mathcal{H}om$ with open immersions (to see which, use [19, 2.4.5.2]) implies that i^*F is i^*A-rigid. It follows then from Theorem 3.1.7(iii) that we may assume $F = A$.

The decomposition $X = \bigsqcup_{s=1}^n U_s$ now yields a decomposition of $F \in \mathsf{D}(X)$:

$$F \simeq \bigoplus_{s=1}^n (i_s)_*(i_s)^*F = \bigoplus_{s=1}^n (i_s)_*(i_s)^*A\,.$$

As before, $(i_s)^*A$ is a semidualizing complex of $\mathcal{O}_{U_s}$-modules, so its support is U_s, and it is indecomposable; see Lemma 1.3.7. Hence $(i_s)_*(i_s)^*A$ is indecomposable, and has support U_s. It then follows from (1.1.2.2) that the decomposition above is orthogonal. Moreover, the complexes $(i_s)_*(i_s)^*A$ are A-rigid; see Definition 3.1.5.

Let $F \simeq F_1 \oplus \cdots \oplus F_r$ be a direct decomposition. It results from Lemma 1.3.7 that this decomposition is orthogonal. Hence $X = \operatorname{Supp}_X F = \bigsqcup_{t=1}^{r} V_t$. Furthermore, $F \in \mathsf{D}^{\mathsf{b}}_{\mathsf{c}}(X) \implies F_t \in \mathsf{D}^{\mathsf{b}}_{\mathsf{c}}(X)$ for all t. Hence $V_t = \operatorname{Supp}_X F_t$ is open and closed; and since F_t is indecomposable, V_t is connected. Thus the V_t are the connected components of X. In particular, $r = n$, and, after renumbering, one may assume $V_t = U_t$ for each t. It remains to observe that $F_s \simeq (i_s)_*(i_s)^*F \simeq (i_s)_*(i_s)^*A$. □

3.2. **Morphisms of rigid complexes.** We present elaborations of Theorem 3.1.7, leading to a simple description of the skeleton of the category of rigid pairs; see Theorem 3.2.6 and Remark 3.2.7.

The result below involves the $\mathrm{H}^0(X, \mathcal{O}_X)$ action on $\mathsf{D}(X)$ described in 1.2.

Theorem 3.2.1. *If (F,ρ), (F',ρ') are A-rigid pairs with $\operatorname{Supp}_X F = \operatorname{Supp}_X F'$ then there exists a unique isomorphism $(F,\rho) \xrightarrow{\sim} (F',\rho')$. In particular, any A-rigid pair (F,ρ) admits a unique isomorphism into a U-canonical one, for some open-and-closed $U \subseteq X$, necessarily the support of F.*

Moreover, if $F' = F$ then with $U_F := \operatorname{Supp}_X F$, there is a unique unit u in the ring $\mathrm{H}^0(U_F, \mathcal{O}_{U_F})$ such that $\rho' = \rho\bar{u}$, where $\bar{u} \in \mathrm{H}^0(X, \mathcal{O}_X)$ is u extended by 0, and the unique isomorphism $(F,\rho) \xrightarrow{\sim} (F,\rho')$ is multiplication in F by $\bar{u}$.

For any endomorphism ϕ of the A-rigid pair (F,ρ) there is a uniquely determined idempotent $u \in \mathrm{H}^0(U_F, \mathcal{O}_{U_F})$ such that ϕ is multiplication by $\bar{u}$.

Proof. Modulo Theorem 3.1.7, the proof is basically that of [5, 7.3]. Indeed, Theorem 3.1.7(iii) implies that we may assume $F = F'$, and that furthermore, we may replace X by U, i.e., assume $F = A$ (so that $\bar{u} = u$).

Each endomorphism of F is multiplication by a unique element u in $\mathrm{H}^0(X, \mathcal{O}_X)$. From Lemma 1.2.1 it follows that multiplication by u induces multiplication by u on $F^\dagger$ and multiplication by u^2 on $\mathsf{R}\mathcal{H}om_X(F^\dagger, F)$. With u_F, resp. u_H, multiplication by u on F, resp. on $\mathsf{R}\mathcal{H}om_X(F^\dagger, F)$, we have then that $u_H\rho = \rho u_F$, see 1.2, so that $u_H^2\rho = u_H\rho u_F = \rho u_F^2$.

In view of this identity, one gets that u_F is an isomorphism from the rigid pair (F,ρ) to the rigid pair $(F,\rho') \iff \rho' u_F = u_H^2 \rho \iff \rho' u_F = \rho u_F^2 \iff \rho' = \rho u_F$. Thus the sought-after u is the unique one such that u_F is the automorphism $\rho^{-1}\rho'$.

In the same vein, when u_F induces an endomorphism of the rigid pair (F,ρ) one gets a relation $\rho u = \rho u^2$, whence, ρ being an isomorphism, $u^2 = u$. □

Corollary 3.2.2. *For any A-rigid complex F, the group of automorphisms of F acts faithfully and transitively on the set of rigidifying isomorphisms ρ of F.* □

Corollary 3.2.3. *If X is connected then every nonzero morphism of A-rigid pairs is an isomorphism.* □

Definition 3.2.4. For any $\mathsf{D}(X)$-map $\phi\colon F \to F'$ of A-rigid pairs, $\operatorname{Supp}_X \phi$ is the union of those connected components of X to which the restriction of ϕ is nonzero.

By Corollary 3.2.3, if X is connected then nonzero maps of A-rigid pairs are isomorphisms. So for a composable pair (ϕ,ψ) of maps of A-rigid pairs,

$$\operatorname{Supp}_X(\phi\psi) = \operatorname{Supp}_X \phi \cap \operatorname{Supp}_X \psi. \tag{3.2.4.1}$$

Corollary 3.2.5. *Let (F,ρ) and (F',ρ') be A-rigid pairs.*

(1) *Suppose that $\operatorname{Supp}_X F \subseteq \operatorname{Supp}_X F'$. Then there is a unique monomorphism $(F,\rho)\hookrightarrow(F',\rho')$ and a unique epimorphism $(F',\rho')\twoheadrightarrow(F,\rho)$.*

(2) *For any morphism $\phi\colon (F,\rho)\to(F',\rho')$, if (G,σ) is an A-rigid pair with $\operatorname{Supp}_X G=\operatorname{Supp}_X\phi$ then ϕ factors uniquely as*
$$(F,\rho)\overset{\phi'}{\twoheadrightarrow}(G,\sigma)\overset{\phi''}{\hookrightarrow}(F',\rho')$$
with ϕ' an epimorphism and ϕ'' a monomorphism.

Thus ϕ is uniquely determined by its source, target and support.

Proof. Looking at connected components separately, one reduces to where X is connected; the assertions then follow from Corollary 3.2.3 and Theorem 3.2.1. □

Here is a *structure theorem* for the category $\mathsf{Rp}_A(X)$ of A-rigid pairs.

Theorem 3.2.6. *Let $\mathsf{OC}(X)$ be the category whose objects are the open-and-closed subsets of X, and whose maps $U\to V$ are the open-and-closed subsets of $U\cap V$, the composition of $S\subseteq U\cap V$ and $T\subseteq V\cap W$ being $S\cap T\subseteq U\cap W$.*

Let $\Psi\colon \mathsf{Rp}_A(X)\to\mathsf{OC}(X)$ be the functor taking $(F,\rho)\in\mathsf{Rp}_A(X)$ to $\operatorname{Supp}_X F$, and taking a morphism $\phi\in\mathsf{Rp}_A(X)$ to $\operatorname{Supp}_X\phi$ (see (3.2.4.1)).

This Ψ is an equivalence of categories.

Proof. Let (F,ρ) and (F',ρ') be A-rigid pairs, $U:=\operatorname{Supp}_X F$, $V:=\operatorname{Supp}_X F'$, and S an open-and-closed subset of $U\cap V$. It follows from Corollary 3.2.5, with (G,σ) the S-canonical pair, that there is a unique map of A-rigid pairs $\phi\colon(F,\rho)\to(F',\rho')$ such that $\operatorname{Supp}_X\phi=S$, whence the conclusion. □

Remark 3.2.7. A quasi-inverse Φ of Ψ can be constructed as follows:

$\Phi\colon\mathsf{OC}(X)\to\mathsf{Rp}_A(X)$ takes an open-and-closed $U\subseteq X$ to an arbitrarily chosen rigid pair (F,ρ) with $\operatorname{Supp}_X F=U$; and then for any $\mathsf{OC}(X)$-map $S\subseteq U\cap V$, $\Phi(S)$ is the unique epimorphism $\Phi U\twoheadrightarrow\Phi S$ followed by the unique monomorphism $\Phi S\hookrightarrow\Phi V$ (see Corollary 3.2.5).

That this describes a functor is, modulo (3.2.4.1), straightforward to see.

Taking into account that the map $S\subseteq U\cap V$ factors as a split epimorphism (namely $S\subseteq U\cap S$) followed by a split monomorphism (namely $S\subseteq S\cap V$), and that any functor respects left and right inverses, one sees that in fact *all* quasi-inverses of Ψ have the preceding form.

In particular, there is a canonical Φ, associating to each U the U-canonical pair. Thus $\mathsf{OC}(X)$ is *canonically isomorphic* to the category of canonical A-rigid pairs.

The next result is in preparation for establishing a gluing property for rigid pairs.

Lemma 3.2.8. *If $g\colon Z\to X$ is a perfect map and F is an A-rigid complex in $\mathsf{D}(X)$, then $\mathsf{L}g^*A\in\mathsf{D}^{\mathsf{b}}_{\mathsf{c}}(Z)$ is semidualizing and $\mathsf{L}g^*F$ is $\mathsf{L}g^*A$-rigid.*

Proof. That $\mathsf{L}g^*A$ is semidualizing is given by Corollary 2.2.6.

If ρ is an A-rigidifying isomorphism for $F\in\mathsf{D}(X)$, then, abusing notation, we let $\mathsf{L}g^*\rho$ be the composed isomorphism
$$\begin{aligned}\mathsf{L}g^*F &\xrightarrow{\ \sim\ } \mathsf{L}g^*\mathsf{R}\mathcal{H}om_X(F^\dagger,F)\\ &\xrightarrow{\ \sim\ } \mathsf{R}\mathcal{H}om_Z(\mathsf{L}g^*F^\dagger,\mathsf{L}g^*F)\\ &\xrightarrow{\ \sim\ } \mathsf{R}\mathcal{H}om_Z(\mathsf{R}\mathcal{H}om_Z(\mathsf{L}g^*F,\mathsf{L}g^*A),\mathsf{L}g^*F),\end{aligned}$$

where the first isomorphism is the result of applying the functor $\mathsf{L}g^*$ to ρ, and the other two come from (2.1.7.1). Thus $\mathsf{L}g^*\rho$ is $\mathsf{L}g^*A$-rigidifying for $\mathsf{L}g^*F$. $\square$

Theorem 3.2.9. *Let $g\colon Z \to X$ be a faithfully flat scheme-map, $W := Z \times_X Z$, $\pi_1\colon W \to Z$ and $\pi_2\colon W \to Z$ the canonical projections.*

Let $A \in \mathsf{D}(X)$ be semidualizing. If (G, σ) is a g^*A-rigid pair such that there exists an isomorphism $\pi_1^*G \simeq \pi_2^*G$, then there is, up to unique isomorphism, a unique A-rigid pair (F, ρ) such that $(G, \sigma) \simeq (g^*F, g^*\rho)$.

Proof. (Uniqueness.) If $g^*F \simeq g^*F'$ then, since

$$g^{-1}\operatorname{Supp}_X F = \operatorname{Supp}_Z g^*F = \operatorname{Supp}_Z g^*F' = g^{-1}\operatorname{Supp}_X F',$$

and g is surjective, therefore $\operatorname{Supp}_X F = \operatorname{Supp}_X F'$; and so by Theorem 3.2.1, there is a unique isomorphism $(F, \rho) \xrightarrow{\sim} (F', \rho')$.

(Existence.) In view of Theorem 3.1.7, we may assume that $G = Jg^*A$ for some idempotent $\mathcal{O}_Z$-ideal J. Then, for $i = 1, 2$, Corollaries C.4 and C.7 yield

$$\begin{aligned}\operatorname{Supp}_W \pi_i^*G &= \operatorname{Supp}_W(\pi_i^*J \otimes^{\mathsf{L}}_W \pi_i^*g^*A)\\ &= \operatorname{Supp}_W \pi_i^*J \cap \operatorname{Supp}_W \pi_i^*g^*A\\ &= \operatorname{Supp}_W \pi_i^*J.\end{aligned}$$

So π_1^*J and π_2^*J, being isomorphic to idempotent ideals with the same support, must be isomorphic. Hence by Proposition C.8, there is a unique idempotent $\mathcal{O}_X$-ideal I such that $J = I\mathcal{O}_Z$. If $F = IA$ then $G \simeq g^*F$.

Let ρ be a rigidifying isomorphism for F, so that $(g^*F, g^*\rho)$ is a g^*A-rigid pair. By Theorem 3.2.1, there is a unique isomorphism $(g^*F, g^*\rho) \xrightarrow{\sim} (G, \sigma)$. $\square$

3.3. **Relative rigidity.** With reference to a G-perfect map $f\colon X \to Y$, we take particular interest in those complexes that are $f^!\mathcal{O}_Y$-rigid—complexes we will simply call *f-rigid*.

For g any essentially étale map (so that, by Proposition 2.5.2, fg is G-perfect), there is a natural isomorphism of functors $(fg)^! \simeq g^*f^!$ (see §B.3). By Lemma 3.2.8, if P is f-rigid then g^*P is (fg)-rigid.

The following *étale gluing* result (where for simplicity we omit mention of rigidifying isomorphisms) is an immediate consequence of Theorem 3.2.9.

Proposition 3.3.1. *Let $Z \xrightarrow{g} X \xrightarrow{f} Y$ be scheme-maps, where f is G-perfect and g is essentially étale and surjective. Let $W := Z \times_X Z$, with canonical projections $\pi_1\colon W \to Z$ and $\pi_2\colon W \to Z$. If P is an (fg)-rigid complex such that there exists an isomorphism $\pi_1^*P \simeq \pi_2^*P$, then there exists, up to isomorphism, a unique f-rigid complex F with $g^*F \simeq P$.* $\square$

Fix a semidualizing complex A on a scheme X. The main result in this section, Theorem 3.3.2, is that for any additive functor from A-rigid complexes to the derived category of some scheme, that takes A to a semidualizing complex C—and hence, by Theorem 3.1.7(iv), takes A-rigid complexes to C-rigid complexes—there is a unique lifting to the category of A-rigid pairs that takes the canonical pair (A, ρ^A) to (C, ρ^C), provided that the functor "respects intersection of supports."

From Theorem 3.3.2 we will derive the behavior of relatively rigid complexes with respect to perfect maps (Corollaries 3.3.4 and 3.3.5). These results generalize—and were inspired by—results in [24, Sections 3 and 6].

Let $\mathsf{Rc}_A(X) \subseteq \mathsf{D}(X)$ be the full subcategory of A-rigid complexes, and let $\mathsf{Rp}_A(X)$ de the category of A-rigid pairs. Let $\varphi_X : \mathsf{Rp}_A(X) \to \mathsf{D}(X)$ be the functor taking (F, ρ) to $F \in \mathsf{Rc}_A(X)$. The rigid pair (A, ρ^A) is defined in Example 3.1.3.

Theorem 3.3.2. *Let X and Z be schemes, let $A \in \mathsf{D}(X)$ be semidualizing, and let $\mathsf{F}\colon \mathsf{Rc}_A(X) \to \mathsf{D}(Z)$ be an additive functor such that $\mathsf{F}A$ is semidualizing.*

There exists at most one functor $\bar{\mathsf{F}}\colon \mathsf{Rp}_A(X) \to \mathsf{Rp}_{\mathsf{F}A}(Z)$, such that

$$\varphi_Z \bar{\mathsf{F}} = \mathsf{F}\varphi_X \quad \textit{and} \quad \bar{\mathsf{F}}(A, \rho^A) = (\mathsf{F}A, \rho^{\mathsf{F}A})\,.$$

For such an $\bar{\mathsf{F}}$ to exist it is necessary that for any idempotent $\mathcal{O}_X$-ideals I, J,

$$\operatorname{Supp}_Z \mathsf{F}(IJA) = \operatorname{Supp}_Z \mathsf{F}(IA) \cap \operatorname{Supp}_Z \mathsf{F}(JA), \tag{3.3.2.1}$$

and it is sufficient that (3.3.2.1) hold whenever $IJ = 0$.

Remark 3.3.3. Let $a, b \in \mathrm{H}^0(X, \mathcal{O}_X)$ be the idempotents such that $I = a\mathcal{O}_X$ and $J = b\mathcal{O}_X$. Since IA admits a monomorphism into A, therefore $\mathsf{F}(IA)$ admits a monomorphism into $\mathsf{F}A$, and it follows from Theorem 3.1.7 that there is a unique idempotent $f(a) \in \mathrm{H}^0(Z, \mathcal{O}_Z)$ with $\mathsf{F}(IA) \simeq f(a)\mathsf{F}A$. By (1.1.2.2), Corollary C.4, and the fact that a semidualizing complex on a scheme is supported at every point of the underlying space, see Lemma 1.3.7, condition (3.3.2.1) amounts then to $f(ab) = f(a)f(b)$.

Before proving Theorem 3.3.2, we gather together some examples. Part (1) of the next corollary elaborates Lemma 3.2.8.

Recall that if $g\colon Z \to X$ is perfect then both $\mathsf{L}g^*B$ and $g^!A$ are semidualizing; see Corollary 2.2.6. If $L \in \mathsf{D}(X)$ is invertible then $L \otimes^{\mathsf{L}}_X A$ is semidualizing, by Corollary 1.5.4(3); and if $F \in \mathsf{D}^+_{\mathsf{qc}}(X)$, then there is as in (2.1.5.2) a natural isomorphism $g^!L \otimes^{\mathsf{L}}_Z \mathsf{L}g^*F \xrightarrow{\sim} g^!(L \otimes^{\mathsf{L}}_X F)$.

Corollary 3.3.4. *Let $g\colon Z \to X$ be a perfect map, and $A \in \mathsf{D}^{\mathsf{b}}_{\mathsf{c}}(X)$ semidualizing.*

(1) *There is a unique functor $g^{**}\colon \mathsf{Rp}_A(X) \to \mathsf{Rp}_{\mathsf{L}g^*A}(Z)$ such that*
$$\varphi_Z\, g^{**} = \mathsf{L}g^* \quad \text{and} \quad g^{**}(A, \rho^A) = (\mathsf{L}g^*A, \rho^{\mathsf{L}g^*A}).$$

(2) *There is a unique functor $g^{!!}\colon \mathsf{Rp}_A(X) \to \mathsf{Rp}_{g^!A}(Z)$ such that*
$$\varphi_Z\, g^{!!} = g^! \quad \text{and} \quad g^{!!}(A, \rho^A) = (g^!A, \rho^{g^!A}).$$

(3) *For each invertible $L \in \mathsf{D}(X)$ there is a unique bifunctor*
$$g^{\otimes}\colon \mathsf{Rp}_{g^!L}(Z) \times \mathsf{Rp}_A(X) \to \mathsf{Rp}_{g^!(L \otimes^{\mathsf{L}}_X A)}(Z)$$
such that
$$\varphi_Z\, g^{\otimes}(P, F) = P \otimes^{\mathsf{L}}_Z \mathsf{L}g^*F$$
and
$$g^{\otimes}\big((g^!L, \rho^{g^!L}), (A, \rho^A)\big) = \big(g^!(L \otimes^{\mathsf{L}}_X A), \rho^{g^!(L \otimes^{\mathsf{L}}_X A)}\big)\,.$$

Proof. Corollary C.7 implies that for either functor, one has in Remark 3.3.3 that $f(a)$ is the image of a under the natural map $\mathrm{H}^0(X, \mathcal{O}_X) \to \mathrm{H}^0(Z, \mathcal{O}_Z)$. Thus $f(ab) = f(a)f(b)$ holds, and so (1) and (2) result from Theorem 3.3.2.

For (3) replace X in Theorem 3.3.2 by the disjoint union $Z \sqcup X$. For $P \in \mathsf{D}(Z)$ and $F \in \mathsf{D}(X)$, let $(P, F) \in \mathsf{D}(Z \sqcup X)$ be the complex whose restriction to Z is P and to X is F. There is an obvious functor $\mathsf{F}\colon \mathsf{D}(Z \sqcup X) \to \mathsf{D}(Z)$ taking

(P, F) to $P \otimes^{\mathsf{L}}_Z \mathsf{L}g^*F$. This functor takes the semidualizing complex $(g^!L, A)$ to the semidualizing complex $g^!L \otimes^{\mathsf{L}}_Z \mathsf{L}g^*A \simeq g^!(L \otimes^{\mathsf{L}}_X A)$. Using (1.1.2.2) and Remark 3.3.3, one verifies that (3.3.2.1) holds; and so (3) results. $\square$

Recall that if $Z \xrightarrow{g} X \xrightarrow{f} Y$ are maps such that g is perfect and f is G-perfect then fg is G-perfect (Proposition 2.5.2). Taking $A = f^!\mathcal{O}_Y$ and $L = \mathcal{O}_X$ in (2) and (3) of Corollary 3.3.4 one gets:

Corollary 3.3.5. *Let $g\colon Z \to X$ be perfect, and $f\colon X \to Y$ G-perfect.*

(1) *If F is f-rigid then $g^!F$ is fg-rigid.*
(2) *If P is g-rigid and F is f-rigid then $P \otimes^{\mathsf{L}}_Z \mathsf{L}g^*F$ is fg-rigid.* $\square$

Corollary 3.3.6. *Let $g\colon Z \to X$ be a proper map such that the natural map is an isomorphism $\mathcal{O}_X \xrightarrow{\sim} \mathsf{R}g_*\mathcal{O}_Z$. Let $A \in \mathsf{D}^+_{\mathsf{qc}}(X)$ be such that $g^!A$ is semidualizing.*

Then A is semidualizing, the canonical map is an isomorphism $\mathsf{R}g_*g^!A \xrightarrow{\sim} A$, and there is a unique functor $g_{**}\colon \mathsf{Rp}_{g^!A}(Z) \to \mathsf{Rp}_A(X)$ such that

$$\varphi_X\, g_{**} = \mathsf{R}g_*\varphi_Z \quad \text{and} \quad g_{**}(g^!A, \rho^{g^!A}) = (\mathsf{R}g_*g^!A, \rho^{\mathsf{R}g_*g^!A}).$$

Hence, if $f\colon X \to Y$ is such that fg is G-perfect then f is G-perfect, and if P is fg-rigid then $\mathsf{R}g_*P$ is f-rigid.

Proof. That A is semidualizing is given by Proposition 2.2.4.

There are, for $E \in \mathsf{D}_{\mathsf{qc}}(X)$, natural isomorphisms, the second from B.3(ii), and the third from (B.1.3),

$$\begin{aligned}
\operatorname{Hom}_{\mathsf{D}(X)}(E, \mathsf{R}g_*g^!A) &\cong \operatorname{Hom}_{\mathsf{D}(Z)}(\mathsf{L}g^*E, g^!A)\\
&\cong \operatorname{Hom}_{\mathsf{D}(X)}(\mathsf{R}g_*(\mathcal{O}_Z \otimes^{\mathsf{L}}_Z \mathsf{L}g^*E), A)\\
&\cong \operatorname{Hom}_{\mathsf{D}(X)}(\mathsf{R}g_*\mathcal{O}_Z \otimes^{\mathsf{L}}_X E, A) \cong \operatorname{Hom}_{\mathsf{D}(X)}(E, A).
\end{aligned}$$

It follows, via [19, 3.4.7(ii)], that the canonical map is an isomorphism

$$\mathsf{R}g_*g^!A \xrightarrow{\sim} A.$$

By assumption, one has the natural isomorphism $\mathrm{H}^0(X, \mathcal{O}_X) \xrightarrow{\sim} \mathrm{H}^0(Z, \mathcal{O}_Z)$. So there is a bijection between the idempotents in these two rings; and also, g is surjective. Hence g^{-1} gives a bijection from the open-and-closed subsets of X to the open-and-closed subsets of Z. Furthermore, for any $P \in \mathsf{D}^{\mathsf{b}}_{\mathsf{c}}(Z)$, $\operatorname{Supp}_Z P$ is closed, whence, g being proper, $U := X \setminus g(\operatorname{Supp}_Z P)$ is open; and the restriction of P to $g^{-1}U$ is acyclic. Thus $\operatorname{Supp}_X \mathsf{R}g_*P \subseteq g(\operatorname{Supp}_Z P)$. The verification of (3.3.2.1), with $\mathsf{F} = \mathsf{R}g_*$ and A replaced by $g^!A$, when $IJ = 0$—so that $\operatorname{Supp}_Z(Ig^!A)$ and $\operatorname{Supp}_Z(Jg^!A)$ are disjoint open-and-closed subsets of Z—is now immediate. The existence and uniqueness of g_{**} follows then from Theorem 3.3.2.

For the last assertion, take $A = f^!\mathcal{O}_Y$. $\square$

Corollary 3.3.7. *Let there be given a tor-independent fiber square* (*see* B.2)

$$\begin{CD}
X' @>{v}>> X\\
@V{h}VV @VV{f}V\\
Y' @>>{u}> Y
\end{CD}$$

in which f is G-perfect.

If u is flat, or if u is perfect and f is proper, then h is G-perfect and for any f-rigid $\mathcal{O}_X$-complex F, $\mathsf{L}v^*F$ is h-rigid.

Proof. Proposition 2.5.9 and [16, p. 245, 3.5.2] imply h is G-perfect and v is perfect. By Corollary 3.3.4(i), $\mathsf{L}v^*F$ is $\mathsf{L}v^*f^!\mathcal{O}_Y$-rigid, i.e., $h^!\mathcal{O}_{Y'}$-rigid; see (2.5.9.1). □

Proof of Theorem 3.3.2. (Uniqueness.) Let (G,σ) be an A-rigid pair.

Set $(\mathsf{F}G,\tau) := \bar{\mathsf{F}}(G,\sigma)$. Let ϕ_G be the unique (split) monomorphism from (G,σ) to the canonical pair (A,ρ^A), so that $\bar{\mathsf{F}}(\phi_G)$ is a (split) monomorphism, necessarily the unique one from $(\mathsf{F}G,\tau)$ to $(\mathsf{F}A,\sigma^{\mathsf{F}A})$, see Corollary 3.2.5. It follows then from Lemma 3.1.6 that τ depends only on F and (G,σ).

Also, for any morphism ϕ of A-rigid pairs, $\varphi_Z\bar{\mathsf{F}} = \mathsf{F}$ implies $\bar{\mathsf{F}}\phi = \mathsf{F}\phi$.

(Necessity of (3.3.2.1)). Let $\Psi_Z\colon \mathsf{Rp}_{\mathsf{F}A}(Z) \to \mathsf{OC}(Z)$ be as in Theorem 3.2.6. Let $\Phi : \mathsf{OC}(X) \to \mathsf{Rp}_A(X)$ be as in Remark 3.2.7, sending an open-and-closed $U \subseteq X$ to I_UA, where I_U is the idempotent $\mathcal{O}_X$-ideal that is $\mathcal{O}_U$ over U and (0) elsewhere. Then $\Psi_Z\bar{\mathsf{F}}\Phi\colon \mathsf{OC}(X) \to \mathsf{OC}(Z)$ respects composition of maps, i.e., (3.3.2.1) holds.

(Existence.) Since any functor preserves a map's property of being split—mono or epi—Theorem 3.1.7(iv) shows that F takes A-rigid complexes to $\mathsf{F}A$-rigid complexes; and the preceding uniqueness argument shows how $\bar{\mathsf{F}}(G,\sigma)$ must be defined. It remains to prove that for any morphism $\phi\colon (G,\sigma) \to (G',\sigma')$ of A-rigid pairs, $\mathsf{F}\phi$ is a morphism of $\mathsf{F}A$-rigid pairs.

Let $U_1,\dots,U_n$ be the connected components of X. For each j, let V_j be the support of the $\mathsf{F}A$-rigid complex $\mathsf{F}(I_{U_j}A)$ (see above). The condition (3.3.2.1), for $IJ = 0$, guarantees that if $j \neq k$ then the open-and-closed subsets V_j and V_k are disjoint. So we need only show that

$(*)$ *the restriction of $\mathsf{F}\phi$ over each V_j is a morphism of $\mathsf{F}A|_{V_j}$-rigid pairs.*

Corollary 3.1.8 shows that $\phi = \sum_{j=1}^n \phi_j$ where for each j, the source and target of ϕ_j each have support that, if not empty, is U_j. Then, since F is additive, $\mathsf{F}\phi = \sum_{j=1}^n \mathsf{F}\phi_j$; and the source and target of $\mathsf{F}\phi_j$ each have support contained in V_j (see the first assertion in Theorem 3.2.1). Hence the restriction of $\mathsf{F}\phi$ over V_j is $\mathsf{F}\phi_j$. Proving $(*)$ is thus reduced to the case where X is connected, so that by Corollary 3.2.3, ϕ is either 0 or an isomorphism.

If $\phi = 0$, $(*)$ is obvious. If ϕ (hence $\mathsf{F}\phi$) is an isomorphism consider the diagram, where $(\mathsf{F}G,\tau) := \bar{\mathsf{F}}(G,\sigma)$, $(\mathsf{F}G',\tau') := \bar{\mathsf{F}}(G',\sigma')$, where $\phi_{G'}$ is as above, and where the maps on the right are induced by those on the left:

$$\begin{tikzcd}
\mathsf{F}G \arrow[r, "\tau"] \arrow[d, "\mathsf{F}\phi"'] & \mathsf{R}\mathcal{H}om_Z(\mathsf{R}\mathcal{H}om_Z(\mathsf{F}G,\mathsf{F}A),\mathsf{F}G) \arrow[d, "\xi"] \\
\mathsf{F}G' \arrow[r, "\tau'"] \arrow[d, "\mathsf{F}\phi_{G'}"'] & \mathsf{R}\mathcal{H}om_Z(\mathsf{R}\mathcal{H}om_Z(\mathsf{F}G',\mathsf{F}A),\mathsf{F}G') \arrow[d, "\xi'"] \\
\mathsf{F}A \arrow[r, "\sigma^{\mathsf{F}A}"'] & \mathsf{R}\mathcal{H}om_Z(\mathsf{R}\mathcal{H}om_Z(\mathsf{F}A,\mathsf{F}A),\mathsf{F}A)
\end{tikzcd}$$

By the above-indicated definition of τ and τ', the bottom square commutes, as does the square obtained by erasing τ'. Since ξ' is a monomorphism, therefore the top square commutes too. Thus $\mathsf{F}\phi$ is a map of $\mathsf{F}A$-rigid pairs. □

Remark 3.3.8. One would naturally like more concrete definitions of the functors in Corollary 3.3.4.

One does find in [24, §3] some explicitly formulated—in DGA terms—versions of special cases of these functors. (Indeed, that's what suggested Corollary 3.3.4.) But getting from here to there does not appear to be a simple matter. One might well have to go via the Reduction Theorem [6, 4.1], the main result of that paper, cf. [5, 8.5.5]); and, say for smooth maps, make use of nontrivial formal properties of Verdier's isomorphism (§B.5).

In Duality Land the well-cultivated concrete and abstract plains are not presently known to be connected other than by forbidding mountain passes, that can only be traversed by hard slogging.

Background

We review background concepts and basic facts having to do with scheme-maps, insofar as needed in the main text. Of special import is the *twisted inverse-image pseudofunctor*, a fundamental object in Grothendieck duality theory.

Rings and schemes are assumed throughout to be noetherian.

Appendix A. Essentially finite-type maps

A.1. A homomorphism $\sigma\colon K\to S$ of commutative rings is *essentially of finite type* if σ can be factored as a composition of ring-homomorphisms

$$K \hookrightarrow K[x_1,\dots,x_d] \to V^{-1}K[x_1,\dots,x_d] \twoheadrightarrow S\,,$$

where $x_1,\dots,x_d$ are indeterminates, $V\subseteq K[x_1,\dots,x_d]$ is a multiplicatively closed set, the first two maps are canonical and the third is surjective. The map σ is *of finite type* if one can choose $V=\{1\}$; the map σ is *finite* if it turns S into a finite (that is, finitely generated) R-module.

A homomorphism $\dot\sigma\colon K\to P$ is (*essentially*) *smooth* if it is flat and (essentially) of finite type, and if for each homomorphism of rings $K\to k$, where k is a field, the ring $k\otimes_K P$ is regular. By [14, 17.5.1], this notion of smoothness is equivalent to the one defined in terms of lifting of homomorphisms.

When $\dot\sigma$ is essentially smooth the P-module $\Omega_{\dot\sigma}$ of relative Kähler differentials is finite projective; we say $\dot\sigma$ has *relative dimension* d if for every $p\in\operatorname{Spec}S$, the free S_p-module $(\Omega_{\dot\sigma})_p$ has rank d.

A.2. A scheme-map $f\colon X\to Y$ is *essentially of finite type* if every $y\in Y$ has an affine open neighborhood $V=\operatorname{Spec}(A)$ such that $f^{-1}V$ can be covered by finitely many affine open sets $U_i=\operatorname{Spec}(C_i)$ so that the corresponding ring homomorphisms $A\to C_i$ are essentially of finite type.

If, moreover, there exists for each i a multiplicatively closed subset $V_i\subseteq A$ such that $A\to C_i$ factors as $A\to V_i^{-1}A \xrightarrow{\sim} C_i$ where the first map is canonical and the second is an isomorphism (in other words, $A\to C_i$ is a *localization* of A), then we say that f is *localizing.* If the scheme-map f is localizing and also set-theoretically injective, then we say that f is a *localizing immersion.*

The map f is *essentially smooth* (of relative dimension d) if it is essentially of finite type and the above data $A\to C_i$ can be chosen to be essentially smooth ring homomorphisms (of relative dimension d). The map f is *essentially étale* if it is essentially smooth of relative dimension 0. Equivalently, f is essentially smooth

(resp. étale) if it is essentially of finite type and formally smooth (resp. étale); see [14, §17.1]. For example, any localizing map is essentially étale.

Remark A.3. We will refer a few times to proofs in [16] that make use of the fact that the diagonal of a smooth map is a quasi-regular immersion. To ensure that those proofs apply here, we note that the same property for *essentially smooth* maps is given by [14, 16.10.2 and 16.9.4].

In [21, 4.1], extending a compactification theorem of Nagata, Nayak shows that *every essentially-finite-type separated map f of noetherian schemes factors as $f = \bar{f}u$ with $\bar{f}$ proper and u a localizing immersion.*

Example A.4. (Local compactification.) A map $f\colon X = \operatorname{Spec} S \to \operatorname{Spec} K = Y$ coming from an essentially finite-type homomorphism of rings $K \to S$ factors as

$$X \xrightarrow{j} W \xhookrightarrow{i} \bar{W} \xrightarrow{\pi} Y,$$

where W is the Spec of a finitely-generated K-algebra T of which S is a localization, j being the corresponding map, where i is an open immersion, and where π is a projective map, so that π is proper and ij is a localizing immersion.

Appendix B. Review of global duality theory

All scheme-maps are assumed to be essentially of finite type and separated.

We recall some global duality theory, referring to [19] and [21] for details.

B.1. To any scheme-map $f\colon X \to Y$ one associates the right-derived direct-image functor $\mathsf{R}f_*\colon \mathsf{D}_{\mathsf{qc}}(X) \to \mathsf{D}_{\mathsf{qc}}(Y)$ and its left adjoint, the left-derived inverse-image functor $\mathsf{L}f^*\colon \mathsf{D}_{\mathsf{qc}}(Y) \to \mathsf{D}_{\mathsf{qc}}(X)$ [19, 3.2.2, 3.9.1, 3.9.2]. These functors interact with the left-derived tensor product $\otimes^{\mathsf{L}}$ via a natural isomorphism

$$\mathsf{L}f^*(M \otimes^{\mathsf{L}}_Y N) \xrightarrow{\ \sim\ } \mathsf{L}f^*M \otimes^{\mathsf{L}}_X \mathsf{L}f^*N \qquad \big(M, N \in \mathsf{D}(Y)\big), \tag{B.1.1}$$

see [19, 3.2.4]; via the functorial map

$$\mathsf{R}f_*F \otimes^{\mathsf{L}}_Y \mathsf{R}f_*G \to \mathsf{R}f_*(F \otimes^{\mathsf{L}}_X G) \qquad \big(F, G \in \mathsf{D}(X)\big) \tag{B.1.2}$$

adjoint to the natural composite map

$$\mathsf{L}f^*(\mathsf{R}f_*F \otimes^{\mathsf{L}}_Y \mathsf{R}f_*G) \xrightarrow{\ \sim\ } \mathsf{L}f^*\mathsf{R}f_*F \otimes^{\mathsf{L}}_X \mathsf{L}f^*\mathsf{R}f_*G \longrightarrow F \otimes^{\mathsf{L}}_X G;$$

and via the *projection isomorphism*

$$\mathsf{R}f_*F \otimes^{\mathsf{L}}_Y M \xrightarrow{\ \sim\ } \mathsf{R}f_*(F \otimes^{\mathsf{L}}_X \mathsf{L}f^*M) \qquad \big(F \in \mathsf{D}_{\mathsf{qc}}(X),\ M \in \mathsf{D}_{\mathsf{qc}}(Y)\big), \tag{B.1.3}$$

defined qua map to be the natural composition

$$\mathsf{R}f_*F \otimes^{\mathsf{L}}_Y M \to \mathsf{R}f_*F \otimes^{\mathsf{L}}_Y \mathsf{R}f_*\mathsf{L}f^*M \to \mathsf{R}f_*(F \otimes^{\mathsf{L}}_X \mathsf{L}f^*M).$$

see [19, 3.9.4)]. The projection isomorphism yields a natural isomorphism

$$\mathsf{R}f_*\mathsf{L}f^*M \simeq \mathsf{R}f_*(\mathcal{O}_X \otimes^{\mathsf{L}}_X \mathsf{L}f^*M) \simeq \mathsf{R}f_*\mathcal{O}_X \otimes^{\mathsf{L}}_Y M. \tag{B.1.4}$$

Interactions with the derived (sheaf-)homomorphism functor $\mathsf{R}\mathcal{H}om$ occur via natural bifunctorial maps:

$$\mathsf{L}f^*\mathsf{R}\mathcal{H}om_Y(M, N) \to \mathsf{R}\mathcal{H}om_X(\mathsf{L}f^*M, \mathsf{L}f^*N) \qquad \big(M, N \in \mathsf{D}(Y)\big)\,, \tag{B.1.5}$$

(see [19, 3.5.6(a)]) which is an *isomorphism* if f is an open immersion [19, p. 190, end of §4.6]; and

$$\mathsf{R}f_*\mathsf{R}\mathcal{H}om_X(F, G) \to \mathsf{R}\mathcal{H}om_Y(\mathsf{R}f_*F, \mathsf{R}f_*G) \qquad \big(F, G \in \mathsf{D}(X)\big), \tag{B.1.6}$$

the latter corresponding via (1.1.1.2) to the natural composition

$$\mathsf{R}f_*\mathsf{R}\mathcal{H}om_X(F,G)\otimes^{\mathsf{L}}_Y\mathsf{R}f_*F\to\mathsf{R}f_*\big(\mathsf{R}\mathcal{H}om_X(F,G)\otimes^{\mathsf{L}}_X F\big)\xrightarrow{\mathsf{R}f_*\varepsilon}\mathsf{R}f_*G,$$

where the first map comes from (B.1.2), and ε is the evaluation map (1.1.1.3).

B.2. For any commutative square of scheme-maps

(B.2.1)
$$\begin{tikzcd} X' \arrow[r, "v"] \arrow[d, "h"'] \arrow[dr, phantom, "\Xi"] & X \arrow[d, "f"] \\ Y' \arrow[r, "u"'] & Y \end{tikzcd}$$

one has the map $\theta_\Xi\colon \mathsf{L}u^*\mathsf{R}f_*\to\mathsf{R}h_*\mathsf{L}v^*$ adjoint to the natural composite map

$$\mathsf{R}f_*\longrightarrow\mathsf{R}f_*\mathsf{R}v_*\mathsf{L}v^*\xrightarrow{\ \sim\ }\mathsf{R}u_*\mathsf{R}h_*\mathsf{L}v^*.$$

When Ξ is a *fiber square* (which means that the map associated to Ξ is an isomorphism $X'\xrightarrow{\ \sim\ }X\times_Y Y'$), and u is *flat*, then θ_Ξ is an *isomorphism.* In fact, for any fiber square Ξ, θ_Ξ *is an isomorphism* $\iff$ Ξ *is tor-independent* [19, 3.10.3].

B.3. Duality theory focuses on the *twisted inverse-image pseudofunctor*

$$f^!\colon \mathsf{D}^+_{\mathsf{qc}}(Y)\to\mathsf{D}^+_{\mathsf{qc}}(X),$$

where "pseudofunctoriality" (also known as "2-functoriality") entails, in addition to functoriality, a family of functorial isomorphisms $c_{g,f}\colon(fg)^!\xrightarrow{\ \sim\ }g^!f^!$, one for each composable pair $Z\xrightarrow{g}X\xrightarrow{f}Y$, satisfying a natural "associativity" property vis-à-vis any composable triple, see, e.g., [19, 3.6.5].

This pseudofunctor is uniquely determined up to isomorphism by the following three properties:

(i) If f is essentially étale then $f^!$ is the usual restriction functor f^*.

(ii) If f is proper then $f^!$ is right-adjoint to $\mathsf{R}f_*$.

(iii) If in a fiber square Ξ as in (B.2.1) the map f (and hence h) is proper and u is essentially étale, then the functorial *base-change map*

(B.3.1) $$\beta_\Xi(M)\colon v^*f^!M\to h^!u^*M\qquad\big(M\in\mathsf{D}^+_{\mathsf{qc}}(Y)\big),$$

defined to be adjoint to the natural composition

$$\mathsf{R}h_*v^*f^!M\ \underset{\theta_\Xi^{-1}}{\xrightarrow{\ \sim\ }}\ u^*\mathsf{R}f_*f^!M\longrightarrow u^*M,$$

is identical with the natural composite *isomorphism*

$$v^*f^!M=v^!f^!M\xrightarrow{\ \sim\ }(fv)^!M=(uh)^!M\xrightarrow{\ \sim\ }h^!u^!M=h^!u^*M.$$

For the existence of such a pseudofunctor, see [21, section 5.2].

B.4. Nayak's theorem [21, 5.3] (as elaborated in [20, 7.1.6]) shows that one can associate, in a unique way, to *every* fiber square Ξ as in (B.2.1) with u (and hence v) flat, a functorial isomorphism

$$\beta_\Xi(M)\colon v^*f^!M\xrightarrow{\ \sim\ }h^!u^*M\qquad\big(M\in\mathsf{D}^+_{\mathsf{qc}}(Y)\big),$$

equal to (B.3.1) when f is proper, and to the natural isomorphism $v^*f^*\xrightarrow{\ \sim\ }h^*u^*$ when f is essentially étale.

B.5. Generalizing (i) in §B.3, let $f\colon X \to Y$ be essentially smooth, so that by [14, 16.10.2] the relative differential sheaf Ω_f is locally free over $\mathcal{O}_X$. On any connected component W of X, the rank of Ω_f is a constant, denoted $d(W)$.

There is a *functorial isomorphism*

$$f^!M \xrightarrow{\sim} \Sigma^d\Omega_f^d \otimes_{\mathcal{O}_X} f^*M \qquad \big(M \in \mathsf{D}_{\mathsf{qc}}(Y)\big),$$

with $\Sigma^d\Omega_f^d$ the complex whose restriction to any W is $\Sigma^{d(W)}\bigwedge_{\mathcal{O}_W}^{d(W)}(\Omega_f\big|_W)$.
(Σ is the usual translation automorphism of $\mathsf{D}(X)$; and $\bigwedge$ denotes "exterior power.")

To prove this, one may assume that X itself is connected, and set $d := d(X)$. Noting that the diagonal $\Delta\colon X \to X \times_Y X$ is defined locally by a regular sequence of length d (see Remark A.3), so that $\Delta^!\mathcal{O}_{X\times_Y X} \otimes^{\mathsf{L}} \mathsf{L}\Delta^*G \cong \Delta^!G$ for all $G \in \mathsf{D}_{\mathsf{qc}}(X\times_Y X)$ [15, p. 180, 7.3], one can imitate the proof of [23, p. 397, Thm. 3], where, in view of (a) above, one can drop the properness condition and take $U = X$, and where finiteness of Krull dimension is superfluous.

B.6. The fact that $\beta_{\Xi}(M)$ in (B.3.1) is an isomorphism for all M whenever u is an open immersion and f is proper, is shown in [19, §4.6, part V] to be equivalent to *sheafified duality,* which is that *for any proper $f\colon X \to Y$, and any $F \in \mathsf{D}_{\mathsf{qc}}(X)$, $M \in \mathsf{D}^+_{\mathsf{qc}}(Y)$, the natural composition, in which the first map comes from* B.1.6,

$$\text{(B.6.1)}\qquad \mathsf{R}f_*\mathcal{H}om_X(F,\, f^!M) \to \mathsf{R}\mathcal{H}om_Y(\mathsf{R}f_*F,\, \mathsf{R}f_*f^!M) \to \mathsf{R}\mathcal{H}om_Y(\mathsf{R}f_*F, M),$$

is an isomorphism.

Moreover, if the proper map f has finite flat dimension, then sheafified duality holds for *all* $M \in \mathsf{D}_{\mathsf{qc}}(Y)$, see [19, 4.7.4].

If f is a *finite* map, then (B.6.1) with $F = \mathcal{O}_X$ determines the functor $f^!$. (See also [11, §2.2].) In particular, if $f\colon \operatorname{Spec} B \to \operatorname{Spec} A$ corresponds to a finite ring homomorphism $A \to B$, and $^\sim$ is the standard sheafification functor, then for an A-complex N, $f^!(N^\sim)$ is the B-complex

$$\text{(B.6.2)}\qquad f^!(N^\sim) = \mathsf{R}\mathrm{Hom}_A(B, N)^\sim,$$

where $\mathsf{R}\mathrm{Hom}_A(B, -)$ denotes the right-derived functor of the functor $\mathrm{Hom}_A(B, -)$ from A-modules to B-modules.

Appendix C. Idempotent ideal sheaves

Definition C.1. Let $(X, \mathcal{O}_X)$ be a local-ringed space, that is, X is a topological space and $\mathcal{O}_X$ is a sheaf of commutative rings whose stalk at each point of X is a local ring (not necessarily noetherian). An $\mathcal{O}_X$-ideal is *idempotent* if it is of finite type (i.e., locally finitely generated) and satisfies the equivalent conditions in the next proposition.

Proposition C.2. *Let $(X, \mathcal{O}_X)$ be a local-ringed space. Consider the following conditions on an $\mathcal{O}_X$-ideal I.*

- (i) *There is an $a \in H^0(X, \mathcal{O}_X)$ such that $a^2 = a$ and $I = a\mathcal{O}_X$.*
- (i′) *The identity map of I extends to an $\mathcal{O}_X$-homomorphism $\pi\colon \mathcal{O}_X \to I$.*
- (ii) *There is an open and closed $U \subseteq X$, with inclusion, say, $i\colon U \hookrightarrow X$, and an $\mathcal{O}_X$-isomorphism $i_*\mathcal{O}_U \simeq I$.*
- (iii) *The $\mathcal{O}_X$-module $\mathcal{O}_X/I$ is flat.*
- (iv) *For all $\mathcal{O}_X$-modules F, the natural map is an isomorphism $I \otimes_X F \xrightarrow{\sim} IF$.*
- (v) *For all $\mathcal{O}_X$-ideals J, $IJ = I \cap J$.*

(vi) $I^2 = I$.

One has the implications

$$\text{(i)} \iff \text{(i}'\text{)} \iff \text{(ii)} \implies \text{(iii)} \iff \text{(iv)} \iff \text{(v)} \implies \text{(vi)};$$

and if I is of finite type then (vi) $\implies$ (i).

Proof. (i) $\Leftrightarrow$ (i$'$). If (i) holds, let π be the map taking $1 \in \mathrm{H}^0(X, \mathcal{O}_X)$ to a. Conversely, given (i$'$), let $a = \pi(1)$.

(ii) $\Rightarrow$ (i). Let a be the global section that is 1 over U and 0 over $X \setminus U$.

(i) $\Rightarrow$ (vi). Trivial.

(vi) $\Rightarrow$ (ii) when I is of finite type (whence (i) $\Rightarrow$ (ii) always). The support of I, $U := \{\, x \in X \mid I_x \neq 0 \,\}$, is closed when I is of finite type. For any $x \in U$, since I_x is a finitely generated $\mathcal{O}_{X,x}$-ideal such that $I_x = I_x^2$, therefore Nakayama's lemma shows that $I_x = \mathcal{O}_{X,x}$. So $X \setminus U = \{\, x \in X \mid \mathcal{O}_{X,x}/I_x \neq 0 \,\}$ is closed, and thus U is open as well as closed. Clearly, $I|_U = \mathcal{O}_U$ and $I|_{X \setminus U} = 0$, whence $I \simeq i_*\mathcal{O}_U$.

(i) $\Rightarrow$ (iii). If (i) holds then the germ of a at any $x \in X$ is 1 or 0, so $(\mathcal{O}/I)_x$ is either (0) or $\mathcal{O}_{X,x}$, both of which are flat over $\mathcal{O}_{X,x}$.

The remaining implications can be tested stalkwise, and so reduce to the corresponding well-known implications for ideals I, J in a local ring R, and R-modules F:

(iii) $\Rightarrow$ (iv). The surjection $I \otimes_R F \twoheadrightarrow IF \subseteq R \otimes_R F$ has kernel $\mathrm{Tor}_1^R(R/I, F) = 0$.

(iv) $\Rightarrow$ (v). $(I \cap J)/IJ$ is the kernel of the natural injective (by (iv)) map

$$R/IJ \cong I \otimes_R R/J \to R \otimes_R (R/J) = R/J.$$

(v) $\Rightarrow$ (iii). Flatness of R/I is implied by injectivity, for all R-ideals J, of the natural map $J/IJ \cong J \otimes_R (R/I) \to R \otimes_R (R/I) = R/I$, with kernel $(I \cap J)/IJ$.

(v) $\Rightarrow$ (vi). Take $J = I$. $\square$

Corollary C.3. (1) *Taking a to $a\mathcal{O}_X$ gives a bijection from the set of idempotent elements of $H^0(X, \mathcal{O}_X)$ to the set of idempotent $\mathcal{O}_X$-ideals.*

(2) *There is a bijection that associates to each idempotent $\mathcal{O}_X$-ideal its support—an open-and-closed subset of X—and to each open-and-closed $U \subseteq X$, with inclusion map i, the unique idempotent $\mathcal{O}_X$-ideal isomorphic to $i_*\mathcal{O}_U$, that is, the ideal whose restriction to U is $\mathcal{O}_U$ and to $X \setminus U$ is* (0). $\square$

Corollary C.4. *A finite-type $\mathcal{O}_X$-ideal I is idempotent if and only if for each $G \in \mathsf{D}(X)$ there exist $\mathsf{D}(X)$-isomorphisms, functorial in G,*

$$\mathsf{R}\mathcal{H}om_X(I, G) \simeq I \otimes^{\mathsf{L}}_X G \simeq IG.$$

Proof. If I is idempotent then over the open set $U := \mathrm{Supp}_X\, I$ one has $I = \mathcal{O}_U$, and over the disjoint open set $X \setminus U$, $I \simeq 0$, so the asserted isomorphisms obviously exist over $X = U \sqcup (X \setminus U)$.

Conversely, if these isomorphisms hold for all members of the natural triangle

$$I \to \mathcal{O}_X \to \mathcal{O}_X/I \xrightarrow{+}$$

then, since $I(\mathcal{O}_X/I) = 0$, application of the functor $\mathsf{R}\mathcal{H}om(I, -)$ yields that the natural map is an isomorphism $I \simeq I^2$ in $\mathsf{D}(X)$, hence in $\mathcal{O}_X$, i.e., $I = I^2$. $\square$

Corollary C.5. *Let X be a locally noetherian scheme. For a complex $L \in \mathsf{D}(X)$ the following conditions are equivalent.*

(i) *L is isomorphic in $\mathsf{D}(X)$ to an idempotent $\mathcal{O}_X$-ideal.*

(ii) *$L \in \mathsf{D}^{\mathsf{b}}_{\mathsf{c}}(X)$ and there exists a $\mathsf{D}(X)$-isomorphism $L \otimes^{\mathsf{L}}_X L \xrightarrow{\sim} L$.*

Proof. If (i) holds then $L \in \mathsf{D}^{\mathsf{b}}_{\mathsf{c}}(X)$ is clear; and taking $G = I$ in C.4, one gets (ii). When (ii) holds, (i) follows easily from [5, 4.9]. □

Proposition C.6. *Let $g\colon Z \to X$ be a morphism of local ringed spaces (so that for each $z \in Z$ the associated stalk homomorphism $\mathcal{O}_{X,gz} \to \mathcal{O}_{Z,z}$ is a local homomorphism of local rings). Let I be an $\mathcal{O}_X$-ideal. If I is idempotent then so is $I\mathcal{O}_Z \cong g^*I \simeq \mathsf{L}g^*I$. The converse holds if g is flat and surjective.*

Proof. If $I = I^2$ then $I\mathcal{O}_Z = (I\mathcal{O}_Z)^2$. Flatness of $\mathcal{O}_X/I$ implies that I is flat and that the natural map $g^*I \to g^*\mathcal{O}_X = \mathcal{O}_Z$ is injective, and thus $\mathsf{L}g^*I \simeq g^*I \cong I\mathcal{O}_Z$.

If g is flat and surjective then for each $x \in X$ there is a $z \in Z$ such that $g(z) = x$, and then there is a flat local homomorphism $\mathcal{O}_{X,x} \to \mathcal{O}_{Z,z}$. Hence if $I\mathcal{O}_Z = (I\mathcal{O}_Z)^2$ then $I_x O_{Z,z} = I_x^2 O_{Z,z}$, i.e., $I_x = I_x^2$. As this holds for all x, therefore $I = I^2$. □

Corollary C.7. *Let $g\colon Z \to X$ be a morphism of local ringed spaces, and I an idempotent $\mathcal{O}_X$-ideal.*

(1) *For any $E \in \mathsf{D}(X)$, there is a unique isomorphism $\mathsf{L}g^*(IE) \simeq I\mathsf{L}g^*E$ whose composition with the natural map $I\mathsf{L}g^*E \to \mathsf{L}g^*E$ is the map obtained by applying $\mathsf{L}g^*$ to the natural map $IE \to E$.*

(2) *If g is a perfect scheme-map then for any $E \in \mathsf{D}^{+}_{\mathsf{qc}}(X)$, there exists a unique isomorphism $g^!(IE) \simeq Ig^!E$ whose composition with the natural map $Ig^!E \to g^!E$ is the map obtained by applying $g^!$ to the natural map $IE \to E$.*

Proof. Uniqueness holds because, $I\mathcal{O}_Z$ being idempotent, $I\mathsf{L}g^*E \simeq I\mathcal{O}_Z \otimes_Z \mathsf{L}g^*E$ is a direct summand of $\mathcal{O}_Z \otimes_Z \mathsf{L}g^*E \simeq \mathsf{L}g^*E$ (Proposition C.2, (iv) and (i′)).

Since both I and $\mathcal{O}_X/I$ are flat over $\mathcal{O}_X$, there are for all $F \in \mathsf{D}(X)$ natural isomorphisms $\mathsf{L}g^*I \otimes^{\mathsf{L}}_Z F \simeq g^*I \otimes_Z F \cong IF$. So for all $E \in \mathsf{D}(X)$,

$$\mathsf{L}g^*(IE) \simeq \mathsf{L}g^*(I \otimes^{\mathsf{L}}_X E) \simeq \mathsf{L}g^*I \otimes^{\mathsf{L}}_Z \mathsf{L}g^*E \simeq I\mathsf{L}g^*E.$$

The composition of these isomorphisms has the property asserted in (1).

Similarly, if g is a perfect scheme-map then, using Theorem 2.1.5, one gets natural isomorphisms for all $E \in \mathsf{D}^{+}_{\mathsf{qc}}(X)$,

$$g^!(IE) \simeq g^!(I \otimes^{\mathsf{L}}_X E) \simeq \mathsf{L}g^*I \otimes^{\mathsf{L}}_Z \mathsf{L}g^*E \otimes^{\mathsf{L}}_Z g^!\mathcal{O}_X \simeq \mathsf{L}g^*I \otimes^{\mathsf{L}}_Z g^!E \simeq Ig^!E,$$

that compose to the isomorphism needed for (2). □

The next result is to the effect that *idempotence satisfies faithfully flat descent* (without any "cocycle condition").

Proposition C.8. *Let $g\colon Z \to X$ be a faithfully flat map, and let $\pi_1\colon Z \times_X Z \to Z$ and $\pi_2\colon Z \times_X Z \to Z$ be the canonical projections. If J is an idempotent $\mathcal{O}_Z$-ideal such that there exists an isomorphism $\pi_1^*J \cong \pi_2^*J$ then there is a unique idempotent $\mathcal{O}_X$-ideal such that $J = I\mathcal{O}_Z$.*

Proof. (Uniqueness.) If $J = I\mathcal{O}_Z = I'\mathcal{O}_Z$ where I and I' are idempotent $\mathcal{O}_X$-ideals with respective supports U and U', then $g^{-1}U = g^{-1}U'$ (both being the support of J), and since g is surjective, therefore $U = U'$, so $I = I'$.

(Existence.) Let V be the support of J. The support of π_1^*J is $\pi_1^{-1}V = V \times_X Z$, and similarly that of π_1^*J is $Z \times_X V$. Hence, since $\pi_1^*J \cong \pi_2^*J$, the following subsets of $Z \times_X Z$ are all the same:

$$V \times_X Z = Z \times_X V = (V \times_X Z) \cap (Z \times_X V) = V \times_X V.$$

If $v \in V$ and $w \in Z$ are such that $g(v) = g(w)$, then there is a field K and a map $\gamma\colon \operatorname{Spec} K \to V \times_X Z = V \times_X V$ such that the set-theoretic images of $\pi_1\gamma$ and $\pi_2\gamma$ are v and w respectively, so $w \in V$. Thus $V = g^{-1}g(V)$.

We claim that $g(V)$ *is open and closed in* X. For this it suffices to show that for each connected component $X' \subseteq X$, $g(V \cap g^{-1}X') = X'$. Without loss of generality, then, we may assume that X is connected, so $X' = X$.

Since g is flat, if $y \in g(V)$ then the generic point x_1 of any irreducible component X_1 of X containing y is also in $g(V)$. In fact $X_1 \subseteq g(V)$, else the preceding argument applied to $\bar{V} := Z \setminus V$ would show that $x_1 \in g(\bar{V}) = X \setminus g(V)$. It results that some open neighborhood of y is in $g(V)$; and thus $g(V)$ is open. Similarly, $g(\bar{V}) = X \setminus g(V)$ is open, so $g(V)$ is closed.

The conclusion follows, with I the idempotent $\mathcal{O}_X$-ideal corresponding to the open-and-closed set $g(V) \subseteq X$. □

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